A FIRST COURSE IN THE DIFFERENTIAL AND INTEGRAL CALCULUS

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NEW YORK
HENRY HOLT AND COMPANY

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December, 1932

PRINTED IN THE
UNITED STATES OF AMERICA

PREFACE

WHILE the present text follows in large measure the lines of development already customary in American texts on the calculus, the following special features may be mentioned:

(1) By actual count there are 2048 examples and exercises. Of these, 179 are worked in full and serve as illustrative material complete in all particulars. Of the remainder, 92 are accompanied by both the answer and a hint as to method of solution, 1154 are accompanied by the answer only, while, finally, 623 have neither answer nor hint given. This unusually large total of varied illustrations and exercises, coupled with the care with which they have been arranged and gradea, should enable the text to meet all requirements in this regard.

(2) The central facts and formulas of the subject are displayed throughout in the form of distinct theorems, each followed by proof. This arrangement enables the student at all times to focus upon the central facts at issue and it is consistent with principles now accepted generally as desirable in the presentation of mathematical facts however elementary or advanced. As to the proofs, it is well known that it is impossible in an ordinary first course in the calculus to furnish strictly rigorous proofs at all points. This is particularly true in dealing with such topics as the definite integral, considered as the limit of a sum, or with the subject of multiple integrals. In such connections the best one can do in a first course is to give what may be termed a "first approximation" to a real proof. This, at least, is all that has been attempted in the present text, the student being told frankly, however, that such is the case. Were it not for the fact that these topics are so exceedingly important in their applications, especially in physics, it would be logical to omit them altogether from the first course, but under the circumstances such an extreme cannot be justified.

(3) Especial attention has been given to the applications of the calculus to physics. In this connection, see in particular the manner in which the derivative, having once been defined, is immediately applied to the determination of rates (pages 16–20); see also the

general physical applications on pages 280–286; 306–316; 334–339. Students, as a rule, find such applications highly attractive because of their novelty and inherent interest, but experience indicates that physical applications in general must be allowed a relatively large amount of time by the instructor if they are to be fully comprehended.

(4) A brief treatment of the method of least squares has been given in the Appendix, the treatment involving the calculus only to the extent of using the simple principles of Chap. IV relative to the maxima and minima of functions. Here again the presentation has been rendered concise and pointed by arranging the material in the form of theorem and proof with abundant exercises supplied. On account of the ever increasing interest of today in the handling of mathematical statistics, a brief discussion of the method of least squares seems in order at present in a calculus text, hence its inclusion here.

(5) The tables at the end of the book include not only a considerable number of the more important standard integrals, but also trigonometric formulas and tables showing the sine, cosine, and tangent, correct to four decimal places, of all angles, expressed in either degrees or radians, at intervals of 10′ from 0° to 90° For convenience in calculations, a table has been added also showing the squares, cubes, square roots, cube roots, and the reciprocals of all integers from 1 to 100.

The author is greatly indebted to the following members of the mathematics department of the University of Michigan who have aided in reading the proofs and, in some cases, in reading the manuscript also: Assistant Professor Norman H. Anning, Dr. Oscar J. Peterson, Mr. Edwin W. Miller, and Mr. Morris Lipcovitz.

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DIFFERENTIAL CALCULUS

CHAPTER I

FUNDAMENTAL NOTIONS AND DEFINITIONS

1. Introduction. The calculus may be regarded as a special but very powerful method for arriving at the central facts of mathematics. The name is derived from the Latin word *calculus*, meaning pebble, it having been the custom of the Romans to use pebbles as an aid in counting, but the science of the calculus as now technically understood and as referred to in this book, did not originate until the 17th century. At that period it was simultaneously but independently discovered by two men, the one an Englishman (Sir Isaac Newton, 1642–1727) and the other a German (Wilhelm Leibnitz, 1646–1716). It was originally called "fluxions" inasmuch as, in contrast to the earlier studies of geometry and algebra, it dealt essentially with fluctuating or variable magnitudes instead of magnitudes that could be regarded as constant or fixed. Naturally the calculus from the beginning made full use of the science of analytic geometry, as this had originated earlier under the French mathematician and philosopher, René Descartes (1596–1650).

In the present chapter we shall consider certain preliminary definitions and ideas which are basal to the subject.

2. Definitions. Quantities which, in any given discussion or problem, are changeable in value are called **variables,** while quantities which do not change are called **constants.**

When the values of one variable, as y, are determined as soon as the values belonging to another variable, as x, are known, the former is said to be a **function** of the latter. Thus, in each of the following expressions, y is a function of x:

$$y = x^2, \quad y = \sqrt{x(x + 2)}, \quad y = \log x, \quad y = 10^x, \quad y = \sin x.$$

Similarly, the area A of a circle whose radius is r may be said to be a function of r, the precise relation between the two variables in this instance being, from elementary geometry,

$$A = \pi r^2.$$

Again, if s represents the distance which a body in falling from rest passes over during the time t, we know that s is a function of t. In particular, if s is measured in feet and t in seconds, and if the resistance of the air be neglected, the functional relation between the two is known from elementary physics to be (approximately)

$$s = 16.1\, t^2.$$

Likewise, a variable may be a function of two or more other variables. Thus each of the expressions

$$xy^2, \quad x^2 + xy + y^2, \quad \log(x^2 + y^2)$$

is a function of the two variables x and y, while each of the expressions

$$x^2y^2z^2, \quad xy + yz + zx, \quad \sqrt{x - y - z}$$

is a function of the three variables x, y, and z.

If, in a given discussion, some variable, say w, is being regarded as a function of one or more variables, say x, y, z, etc., then w is commonly referred to as the **dependent variable,** while x, y, z, etc. are referred to as the **independent variables.** Thus, the area of a rectangle may be regarded as a variable dependent upon the length and breadth, the latter two measurements being the independent variables.

3. Notation of functions. The symbols $F(x)$, $f(x)$, $\phi(x)$, $\psi(x)$,* and the like are commonly used to denote functions of x. Thus the equation

$$y = f(x)$$

is read "y is a function of x." Similarly, the symbols $F(x, y)$, $f(x, y)$, $\phi(x, y)$, $\psi(x, y)$, etc. denote functions of the two variables x and y.

If $f(x)$ is any given function of x, the symbol $f(a)$ is used to denote the value of $f(x)$ when x has the special value a.

Thus, if

$$f(x) = 2x^2 - 3x + 1, \tag{1}$$

then

$$f(4) = 2\ 4^2 - 3 \cdot 4 + 1 = 32 - 12 + 1 = 21.$$

Likewise, if $f(x)$ is as in (1), we have for example

$$f(m) = 2m^2 - 3m + 1,$$

$$f(p + q) = 2(p + q)^2 - 3(p + q) + 1,$$

$$f\left(\frac{a - 1}{a + 1}\right) = 2\left(\frac{a - 1}{a + 1}\right)^2 - 3\left(\frac{a - 1}{a + 1}\right) + 1.$$

---

*ϕ is pronounced fee, ψ is pronounced psee.

EXERCISES

1. In each of the following functional relations find the values of the dependent variable corresponding respectively to the values 0, 1, 2, 3, -1, $-\frac{1}{2}$ of the independent variable:

$$(a)\ y = 2x - 1, \quad (b)\ y = 3x^2, \quad (c)\ q = \frac{3p-1}{2p+3},$$
$$(d)\ s = \sqrt{\frac{t^2+1}{t+2}}, \quad (e)\ w = 4^v, \quad (f)\ \rho = \theta^3 + 3^\theta.$$

2. If $f(x) = 2x^3 + 5x^2 - 11x + 4$, show that

$$f(1) = 0, \quad f(2) = 18, \quad f(3) = 70, \quad f(\tfrac{1}{2}) = 0, \quad f(0) = 4, \quad f(-4) = 0,$$
$$f(x-1) = 2x^3 - x^2 - 15x + 18, \quad f(x^2) = 2x^6 + 5x^4 - 11x^2 + 4,$$
$$f(x+h) = 2x^3 + (6h+5)x^2 + (6h^2 + 10h - 11)x + (2h^3 + 5h^2 - 11h + 4)$$

3. If $f(x) = 2x^3 + 5x$, show that $f(-x) = -f(x)$.

4. If $f(x) = 4x^4 + 3x^2 - 5$, show that $f(-x) = f(x)$.

5. If $F(\theta) = \sin\theta$, find the value of each of the following:

$$F(0), \quad F\left(\frac{\pi}{4}\right), \quad F\left(\frac{\pi}{2}\right), \quad F(\pi).$$

6. Given $F(y) = y^4 - 3y^3 + 1$ and $f(y) = y^2 - 3y + 1$.
Show that $F(0) = f(0)$, $F(3) = f(3)$, $F(\pm 1) = f(\pm 1)$.

7. If $F(x) = \log\frac{1-x}{1+x}$, show that

$$F(x) + F(y) = F\left(\frac{x+y}{1+xy}\right) \quad \text{and} \quad F(x) - F(y) = F\left(\frac{x-y}{1-xy}\right).$$

8. If $\phi(x) = x^2$, show that $\phi(\sin x) + \phi(\cos x) = 1$.
HINT. See page 359.

9. If $f(\theta) = 2\theta^2 - 1$, show that $f(\cos\theta) = \cos 2\theta$.

10. If $F(z) = 2z\sqrt{1-z^2}$, show that $F(\sin z) = F(\cos z) = \sin 2z$.

11. If $\phi(u) = a^u + a^{-u}$, where a is a constant, show that

$$\phi(2u) = [\phi(u)]^2 - \phi(0),$$
$$\phi(3u) = [\phi(u)]^3 - 3\phi(u),$$
$$\phi(u+v)\phi(u-v) = \phi(2u) + \phi(2v).$$

12. If $f(x, y) = x + y$ and $F(x, y) = x^2 - xy + y^2$, show that

$$f(x, y)F(x, y) = f(x^3, y^3)$$

and

$$f(x, -y)F(-x, y) = f(x^3, -y^3).$$

13. If $f(x, y, z) = x^2 + y^2 + z^2$ and $F(x, y, z) = xy + yz + zx$, show that

$$f(x, y, z) + 2F(x, y, z) = [f(\sqrt{x}, \sqrt{y}, \sqrt{z})]^2.$$

14. If $f(x) = \tan x$, show that

$$f(x-y) = \frac{f(x) - f(y)}{1 + f(x)f(y)}.$$

4. The theory of limits. The calculus is based upon what is called in mathematics the "theory of limits," or, in the earlier language of the subject, the "doctrine of limits." We shall now describe the central features of this theory.

If a function $f(x)$ becomes and remains as near as we please to some fixed value A for all values of x that are sufficiently near to some fixed value p, then $f(x)$ is said to approach A as a limit as x approaches p. Symbolically, such a condition on the part of $f(x)$ is expressed by the equation

$$\lim_{x \to p} f(x) = A. \tag{1}$$

Thus, each of the following equations is seen by inspection to be true:

$$\lim_{x \to 0} 3\,x^2 = 0, \quad \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}, \quad \lim_{x \to -1} (x^2 + x) = 0, \quad \lim_{x \to 0} 10^x = 1.$$

A similar situation is that in which a function $f(x)$ becomes and remains as near as we please to some fixed value A for all positive values of x that are sufficiently large. This situation is represented symbolically by the equation

$$\lim_{x \to +\infty} f(x) = A. \tag{2}$$

Thus, by elementary algebra the following are immediately seen to be true equations:

$$\lim_{x \to +\infty} \frac{1}{x} = 0, \quad \lim_{x \to +\infty} \left(\frac{2}{x^2} + 1\right) = 1, \quad \lim_{x \to +\infty} 10^{-x} = 0.$$

An illustration of this type of limit is also afforded in elementary geometry in the fact that the limit approached by the perimeter of a regular polygon inscribed in a circle as the number of sides of the polygon is indefinitely increased is the circumference of the circle.

The following, which are analogous to (1) and (2), are other type-forms of symbolic equations common in the theory of limits:

$$\lim_{x \to -\infty} f(x) = A, \quad \lim_{x \to p} f(x) = +\infty, \quad \lim_{x \to p} f(x) = -\infty, \tag{3}$$
$$\lim_{x \to +\infty} f(x) = +\infty, \quad \lim_{x \to -\infty} f(x) = +\infty, \quad \lim_{x \to +\infty} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = -\infty.$$

The first of these indicates that $f(x)$ becomes and remains as near as we please to the fixed value A for all values of x that are sufficiently large negatively; the second indicates that $f(x)$ becomes and remains larger than any assigned positive number for all values of x that are sufficiently near to the fixed value p, etc.

Thus, by elementary mathematics the following are at once recognized as true equations:

$$\lim_{x \to -\infty} \frac{1}{x} = 0, \qquad \lim_{x \to 0} \frac{1}{x^2} = +\infty, \qquad \lim_{x \to 1} \frac{-5}{(x-1)^2} = -\infty,$$

$$\lim_{x \to +\infty} 10^x = +\infty, \quad \lim_{x \to +\infty} \log x = +\infty, \quad \lim_{x \to -\infty} (3x + 1) = -\infty.$$

Note. In the foregoing considerations it is to be understood that the limit in question may or may not be actually reached in any given case, depending upon additional circumstances. Thus, if a regular polygon of 10 sides be conceived of as inscribed in a circle during the first minute of time, then a polygon of 20 sides be similarly inscribed during the next $\frac{1}{2}$ minute, then a polygon of 40 sides in the next $\frac{1}{4}$ of a minute, a polygon of 80 sides during the next $\frac{1}{8}$ of a minute, and so on, the circumference, or limit, approached by the perimeter will actually be reached in 2 minutes, since the sum of the time periods thus becomes the unlimited geometric progression

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} +$$

and this progression, by elementary algebra, has as its sum *

$$\frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

On the other hand, if the various polygons are formed, for example, at the constant rate of one per minute, evidently the circumference can never be actually attained.

5. Theorems concerning limits.

The following theorems, the truths of which are in large part self-evident, we shall hereafter use as occasion may require.†

Theorem I. *The limit of the sum, or difference, of two variables is equal to the sum, or difference, of their separate limits.*

Thus,

$$\lim_{x \to 0} (3x^2 - 2x + 1) = \lim_{x \to 0} 3x^2 - \lim_{x \to 0} 2x + \lim_{x \to 0} 1 = 0 - 0 + 1 = 1.$$

Theorem II. *The limit of the product of two or more variables is equal to the product of their separate limits.*

Thus,

$$\lim_{x \to \pi} x \sin x = \lim_{x \to \pi} x \lim_{x \to \pi} \sin x = \pi \cdot 0 = 0.$$

---

* See, for example, the author's *College Algebra* (published by Macmillan Co.), page 81.

† Rigorous proofs of these theorems properly belong to more advanced aspects of mathematics, hence are here omitted. It should be noted, however, that the separate limits referred to in the conclusions of the theorems are always assumed to have definite values.

Theorem III. *The limit of the quotient of two variables is equal to the quotient of their separate limits, provided the variable which occurs in the denominator does not have zero as its limit.*

Thus,

$$\lim_{x \to 3} \frac{x^2 + 1}{x - 2} = \frac{\lim\limits_{x \to 3} (x^2 + 1)}{\lim\limits_{x \to 3} (x - 2)} = \frac{10}{1} = 10.$$

In the statement of Theorem III the reason for assuming that the limit of the denominator is not equal to zero is that if this should happen, the quotient of the separate limits would be of the form $\frac{A}{0}$ and would thus become meaningless whatever the value of A, since division by zero is an impossible operation in mathematics.

6. The form $\frac{0}{0}$. If, in using Theorem III of Art. 5, it happens that the quotient of the two separate limits takes the special form $\frac{0}{0}$, thus presenting not only a zero denominator but also a zero numerator, the quotient of the two variables may still approach a determinate limit. To find this limit, however, some preliminary simplifications of the original quotient are usually necessary, as illustrated in the following example:

Example. Determine the value of

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Solution. If Theorem III be applied at once, we obtain

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{\lim\limits_{x \to 1} (x^2 - 1)}{\lim\limits_{x \to 1} (x - 1)} = \frac{0}{0}.$$

This result, for the reasons indicated in Art. 5, is meaningless. However, the limit in question is perfectly determinate as appears from the following steps in which a simpler form for the given quotient is obtained by elementary algebra before the limit is taken

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

The desired value, therefore, is 2.

Similarly, we have

$$\lim_{x \to -1} \frac{x^3 + 1}{x + 1} = \lim_{x \to -1} (x^2 - x + 1) = 3.$$

7. Functions considered graphically. In the calculus, as in analytic geometry, it is customary to speak of the graph of a function $f(x)$ as the curve formed by the locus of all the points whose coordinates (x, y) satisfy the equation

$$y = f(x).$$

Thus, the graph of the function x^2 is the parabola whose equation is

$$y = x^2,$$

the curve in this case having the form and position shown in Fig. 1.

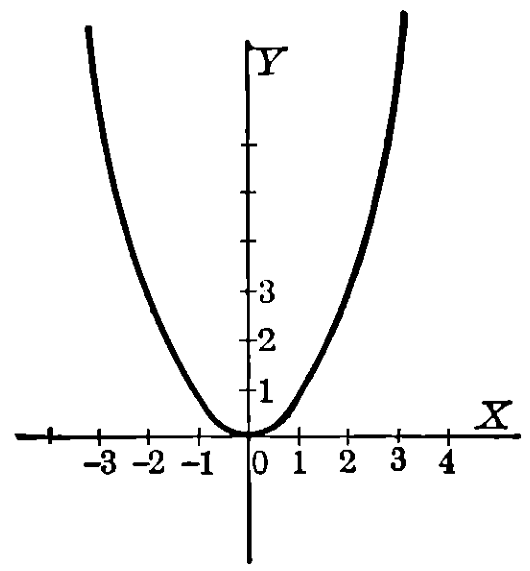

FIG. 1

While many of the graphs considered in analytic geometry are smooth curves throughout their entire extent, as illustrated in a parabola or an ellipse, it is easy to find a function whose graph, when considered at one or more of its points, fails to possess this character. For example, consider the function

$$y = \frac{1}{x}.$$

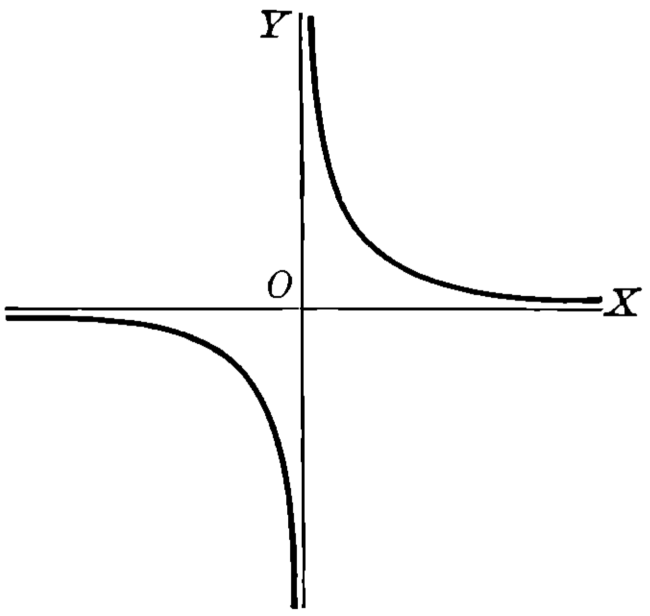

FIG. 2

Here the graph is smooth everywhere except at the point upon it where $x = 0$, as shown in Fig. 2. At this particular point a break occurs. Such a special point is known as a **point of discontinuity.** Moreover, such a point is known as a point of **infinite discontinuity,** since as one approaches it in any manner the curve recedes indefinitely. At such a point the function itself is said to **become infinite.**

On the other hand, if one considers the function

$$y = \frac{1}{1 + 10^{-\frac{1}{x}}}, \quad (1)$$

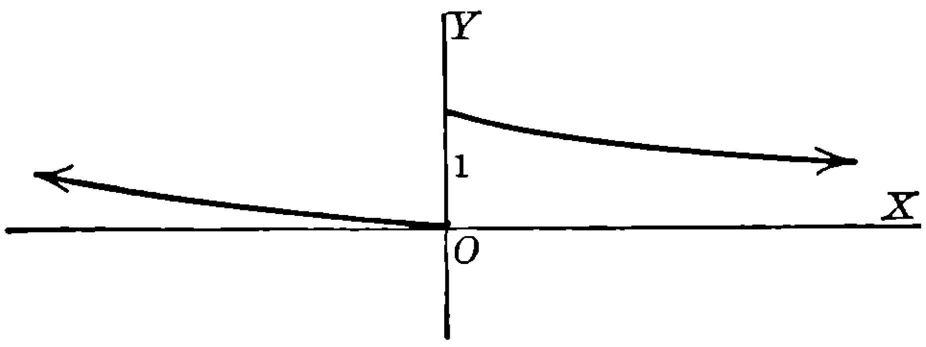

FIG. 3

the graph, as shown in Fig. 3, has a point of discontinuity where $x = 0$, yet it does not recede indefinitely in the neighborhood of this point.

In fact, as x comes closer and closer to zero through *negative* values, y comes closer and closer also to the value zero, since, for such a change in x, the denominator of (1) grows continually larger and larger without limit, thus making the fraction (1) smaller and smaller, but if x approaches zero through *positive* values, the same fraction comes closer and closer to the value

$$\frac{1}{1+0} = 1.$$

Such a point is known as a point of **finite discontinuity.**

Ordinarily, the graph of a function when considered at any one of its points is smooth, as stated above, in which case the curve is said to be **continuous** at the point in question. More precisely, $f(x)$ is continuous at the point upon it where $x = p$ in case $f(p)$ is a determinate number and

$$\lim_{x \to p} f(x) = f(p).$$

We shall be concerned in the present book chiefly with functions which, at least at the points under consideration, are continuous.

EXERCISES

1. State accurately the meaning of each of the equations (3) of Art. 4.

2. Ascertain by inspection the correctness of each of the following equations, giving reasons:

(*a*) $\lim_{x \to 1} (3x^2 - 2x - 1) = 0$;

(*b*) $\lim_{x \to +\infty} \frac{2}{x^2+1} = 0$;

(*c*) $\lim_{x \to -\infty} \frac{2}{x^2+1} = 0$;

(*d*) $\lim_{x \to 1} \frac{4}{(x-1)^2} = +\infty$;

(*e*) $\lim_{x \to +\infty} (x^2 - x - 1) = +\infty$;

(*f*) $\lim_{x \to -\infty} 3x^2 = +\infty$;

(*g*) $\lim_{x \to 0} \frac{1}{\sin^2 x} = +\infty$;

(*h*) $\lim_{x \to +\infty} x \log x = +\infty$.

3. Establish each of the following equations, using the theorems of Art. 5:

(*a*) $\lim_{x \to 2} (2x^2 - x + 1) = 7$;

(*b*) $\lim_{x \to -3} (\frac{2}{3}x^3 + 3x^2 - 5x + 2) = 26$;

(*c*) $\lim_{x \to \pi} (2 \sin x + 3 \cos x) = -3$;

(*d*) $\lim_{x \to 1} (x - 3)(x^2 + 2) = -6$;

(*e*) $\lim_{x \to 2} (x^2 + 1)(x^2 - 1)(x^3 + x) = 150$;

(*f*) $\lim_{t \to -1} \frac{t^2+1}{t+2} = 2$;

(*g*) $\lim_{s \to \frac{\pi}{2}} \frac{\sin s}{s} = \frac{2}{\pi}$;

(*h*) $\lim_{x \to -\pi} \frac{\tan x + \sin x}{\cos x} = 0$;

(*i*) $\lim_{x \to 0} \frac{10^x - 10^{-x}}{2} = 0$;

(*j*) $\lim_{x \to 1} \frac{\log x + \sin \pi x}{x^2} = 0$

4. Show that $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$; also that $\lim\limits_{x \to -2} \dfrac{x^2 - 4}{x + 2} = -4$.

5. Show that $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x^2 - 2x - 3} = \dfrac{5}{4}$.

6. Show that $\lim\limits_{x \to -1} \dfrac{x^2 - 1}{x^3 + 1} = -\dfrac{2}{3}$.

7. By use of familiar trigonometric formulas (see Appendix, page 359) together with the theorems of Art. 5 establish each of the following equations:

(*a*) $\lim\limits_{x \to 0} \dfrac{\sin x}{\tan x} = 1$;

(*b*) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\tan x} = 2$;

(*c*) $\lim\limits_{x \to 0} \dfrac{\tan 2x}{\sin x} = 2$;

(*d*) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos^2 x}{\sin 2x} = 0$.

8. Prove that $\lim\limits_{x \to +\infty} \left(\dfrac{3x + 2}{x}\right) = 3$.

SOLUTION. $\lim\limits_{x \to +\infty} \left(\dfrac{3x + 2}{x}\right) = \lim\limits_{x \to +\infty} \left(3 + \dfrac{2}{x}\right) = 3 + 0 = 3$.

9. Prove that $\lim\limits_{x \to +\infty} \left(\dfrac{2x - 1}{3 - x}\right) = -2$.

HINT. First divide both numerator and denominator by x. Now apply Theorems III and I of Art. 5.

10. Prove that $\lim\limits_{x \to +\infty} \left(\dfrac{2x^2 - x + 1}{3x^2 + 2}\right) = \dfrac{2}{3}$.

HINT. First divide both numerator and denominator by x^2.

11. Prove that $\lim\limits_{x \to +\infty} \dfrac{x - \dfrac{3}{x}}{2x^2 + 1} = 0$.

12. Show that $\lim\limits_{x \to +\infty} \dfrac{3x^3}{x^3 - x^2 + 1} = \lim\limits_{x \to -\infty} \dfrac{3x^3}{x^3 + x^2 + 4}$.

13. State the point, or points, of discontinuity belonging to each of the following functions, and determine in each instance whether it is a point of infinite or of finite discontinuity:

(*a*) $\dfrac{2}{x - 2}$;

(*b*) $\dfrac{x + 1}{x^2 - 4}$;

(*c*) $3x + \dfrac{5}{x^2}$;

(*d*) $\dfrac{1}{\sin \pi x}$;

(*e*) $\tan \pi x$;

(*f*) $\tan \dfrac{\pi}{x}$.

14. Show that the curves $y = \sin x$ and $y = \cos x$ are continuous at every point.

15. Evaluate each of the following limits:

(*a*) $\lim\limits_{x \to -1} \dfrac{3x^2 - x + 1}{4x^3 + 1}$;

(*b*) $\lim\limits_{x \to 2} (x^2 + 2)^2$;

(*c*) $\lim\limits_{x \to 0} x\, 10^{-x}$;

(*d*) $\lim\limits_{x \to -\frac{1}{2}} \dfrac{4x^2 - 1}{2x + 1}$;

(*e*) $\lim\limits_{x \to 1} \dfrac{1 - x}{\sqrt{1 - x^2}}$;

(*f*) $\lim\limits_{x \to 1} \dfrac{1 - x^3}{1 - x^2}$;

(*g*) $\lim\limits_{x \to 1} x^2 \log x$;

(*h*) $\lim\limits_{x \to -1} x \log x^2$;

(*i*) $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{1 - \cos x}$;

(*j*) $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{1 + \cos x}$;

(*k*) $\lim\limits_{x \to \frac{\pi}{2}} \log \sin x$;

(*l*) $\lim\limits_{x \to \pi} 10^{\sin x}$.

CHAPTER II

THE DERIVATIVE OF A FUNCTION

8. Definitions. The student will recall from analytic geometry that the slope of a straight line is defined to be the tangent of the angle which the line makes with the positive direction of the X-axis, as the angle XQP in Fig. 4.

This being understood, let the curve in Fig. 5(a) (or 5(b)) represent the graph of the equation

$$y = f(x), \tag{1}$$

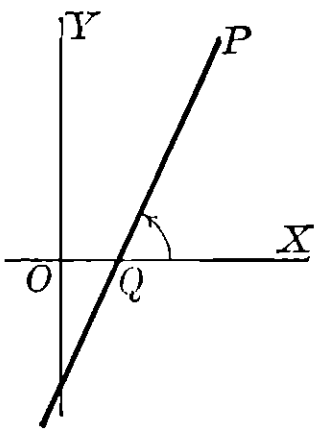

FIG. 4

where $f(x)$ is a given function of x, and let $P = (x, y)$ be any point upon this curve; also let QPT be the tangent line to the curve at P

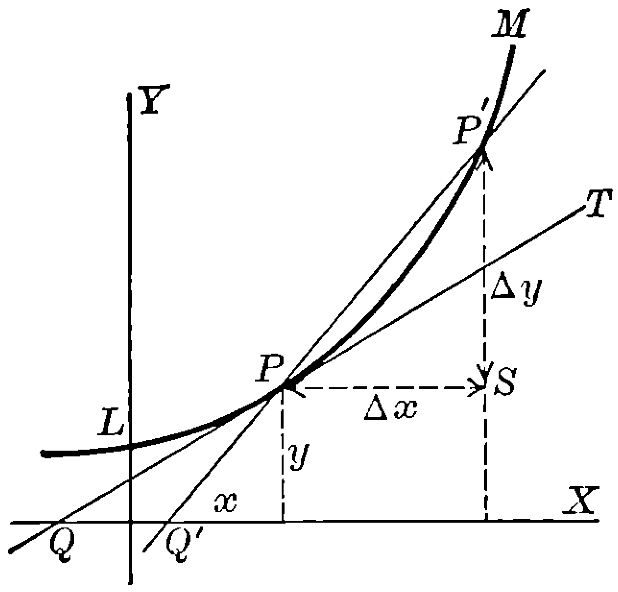

FIG. 5(a)

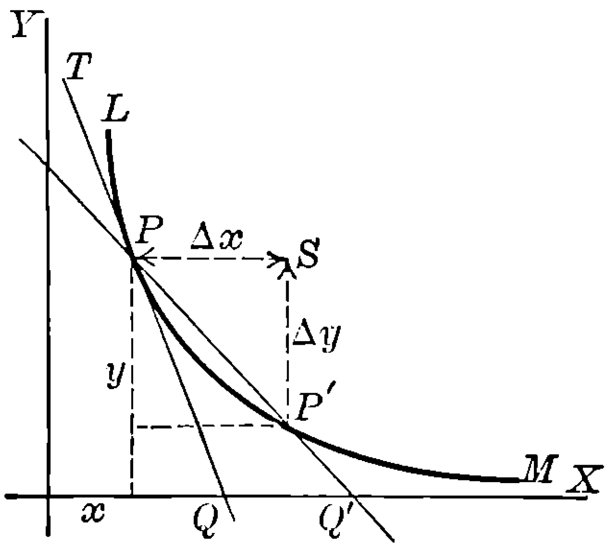

FIG. 5(b)

We shall now show how to determine an expression, known as the **derivative** of $f(x)$ and represented by the symbol $f'(x)$, which measures the slope of this tangent line from point to point along the curve (1).

9. Determination of derivative. Regarding the point $P = (x, y)$ at first as fixed in position, let us assign to its abscissa x a small change, or increment, representing this by the symbol Δx (read delta x, not delta times x, and regarded for the present as positive). Denote, by Δy the increment which y then takes in order that the point $P' = (x + \Delta x, y + \Delta y)$ shall also lie upon the curve. In either Fig. 5(a)

or Fig. 5(b) this increment Δy is represented by the line SP', but in Fig. 5(a) it is positive while in Fig. 5(b) it is negative, inasmuch as in this case the ordinate of P' is less than that of P. For the given values of x and y we may then write in either case, as a result of (1),

$$y = f(x) \tag{2}$$

and

$$y + \Delta y = f(x + \Delta x). \tag{3}$$

Subtracting (2) from (3), member by member, we obtain

$$\Delta y = f(x + \Delta x) - f(x),$$

or, dividing both members by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{4}$$

Now the ratio $\frac{\Delta y}{\Delta x}$ here appearing, when interpreted in Fig. 5(a), is seen to be the tangent of the angle SPP', which in turn is equal to the tangent of the angle $XQ'P$ When likewise interpreted in Fig. 5(b), the same ratio $\frac{\Delta y}{\Delta x}$ is seen to be the negative of the tangent of the angle SPP' (Δy being now negative) and hence it is equal to the tangent of the obtuse angle $XQ'P$, this angle being the supplement of SPP' In either case, then, $\frac{\Delta y}{\Delta x}$ is the slope of the secant line PP' But, as we employ a smaller and smaller value for Δx in the foregoing statements, the corresponding value of Δy lies nearer and nearer to zero, while the point P' lies closer and closer to the fixed point P; also Q' lies closer and closer to Q, so that the slope of the secant line comes nearer and nearer to the slope of the tangent line at P, approaching the latter value in fact as a limit when Δx approaches zero. The desired expression, or derivative of the function $f(x)$, which measures the slope of the tangent to the curve $y = f(x)$ at P is therefore given in all cases by the general formula *

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{5}$$

---

* We here assume that the limit (5) actually exists. There are exceptional cases in which this is not the case even though the curve $y = f(x)$ may be continuous at P, but we shall in no wise be concerned with such cases in the present text.

Besides the symbol $f'(x)$ used as above to denote the derivative of a given function $y = f(x)$, it is customary to use also, as may be desired, any one of the symbols y', $D_x y$, or $\frac{dy}{dx}\cdot$ In all cases, the process of finding the derivative, that is, of evaluating the limit (5), is called **differentiating** the function.

Hereafter the slope of the tangent line to a curve at any point P upon it will be briefly referred to as the **slope of the curve** at P.

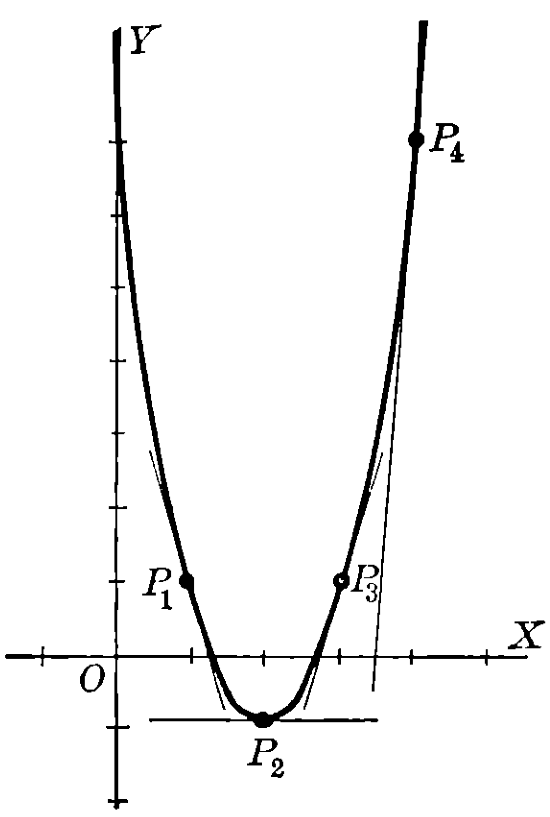

FIG. 6

Example 1. Find the derivative of the function

$$y = 2\,x^2 - 8\,x + 7 \qquad (6)$$

and thus determine the slope of its graph at the points on the curve whose abscissas are respectively $x = 1$, $x = 2$, $x = 3$, and $x = 4$.

SOLUTION. Here

$$f(x) = 2\,x^2 - 8\,x + 7$$

so that the general formula (5) gives

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2(x + \Delta x)^2 - 8(x + \Delta x) + 7 - (2\,x^2 - 8\,x + 7)}{\Delta x}. \end{aligned}$$

Upon simplifying the numerator by expanding $(x + \Delta x)^2$, removing parentheses, and observing mutual cancellations of terms, we obtain

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{4\,x\,\Delta x - 8\,\Delta x + 2\,\overline{\Delta x}^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (4\,x - 8 + 2\,\Delta x) = 4\,x - 8. \end{aligned}$$

The desired derivative of the function (6) is therefore

$$f'(x) = 4\,x - 8. \qquad (7)$$

It follows that at the special point P_1 (see Fig. 6), whose abscissa is $x = 1$, the slope of the curve is equal to $4 \;\, 1 - 8 = -4$. Similarly, at the point P_2, having the abscissa $x = 2$, the slope is seen from (7) to be $4 \;\, 2 - 8 = 0$, thus indicating that the tangent to the curve at this point is parallel to the X-axis. Likewise, at the points P_3 and P_4, where the abscissas are $x = 3$ and $x = 4$, the slopes are equal respectively to $4 \;\, 3 - 8 = 4$ and $4 \;\, 4 - 8 = 8$.

Example 2. Differentiate the function

$$y = x^3$$

and thus determine the slope of its graph at the points having the abscissas $x = -1$, $x = 0$, and $x = 1$.

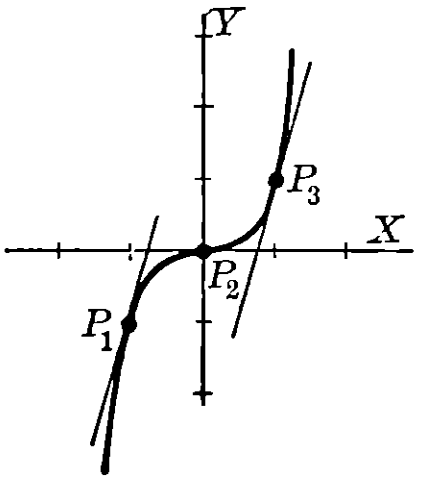

FIG. 7

SOLUTION. We have

$$f(x) = x^3$$

so that, by formula (5),

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3\,x^2\,\Delta x + 3\,x\,\overline{\Delta x}^2 + \overline{\Delta x}^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} (3\,x^2 + 3\,x\,\Delta x + \overline{\Delta x}^2) = 3\,x^2. \end{aligned}$$

The desired derivative, therefore, is

$$f'(x) = 3\,x^2.$$

The slopes of the curve at the special points P_1, P_2, P_3, having the abscissas $x = -1$, $x = 0$, and $x = 1$, are therefore equal respectively to

$$3\ (-1)^2 = 3,\ 3\ 0^2 = 0, \text{ and } 3\ 1^2 = 3.$$

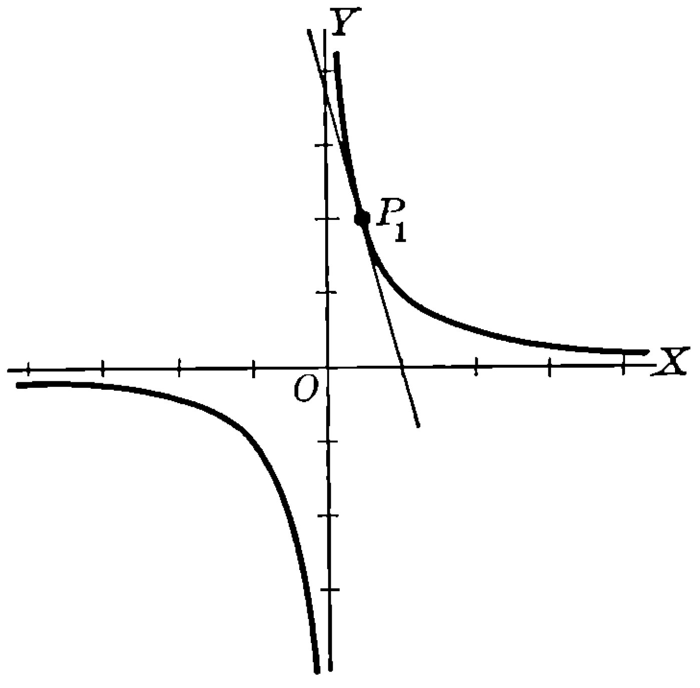

FIG. 8

Example 3. Differentiate the function

$$y = \frac{1}{x}$$

and thus determine the slope of its graph at the point upon it whose abscissa is $x = \frac{1}{2}$.

SOLUTION. We have by formula (5)

$$\begin{aligned} y' &= \lim_{\Delta x \to 0} \left(\frac{1}{x + \Delta x} - \frac{1}{x}\right) \cdot \frac{1}{\Delta x} \\ &= \lim_{x \to 0} \left(\frac{-\Delta x}{x(x + \Delta x)}\right) \frac{1}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(\frac{-1}{x(x + \Delta x)}\right) = -\frac{1}{x^2}. \end{aligned}$$

At the particular point P_1 having the abscissa $x = \frac{1}{2}$ the slope is therefore equal to $-1/(\frac{1}{2})^2 = -4$.

Note. The student should observe that formula (5), when applied to a given function $f(x)$, leads in the limit to the meaningless form $\frac{0}{0}$ (see Art. 6). However, a decisive answer may usually be secured, as in the foregoing examples, by making suitable simplifications before allowing Δx to approach zero. This procedure should be followed in working the following exercises.

EXERCISES

1. Find the derivative of the function $f(x) = x^2 - 2x + 2$.

Ans. $f'(x) = 2x - 2$.

2. By means of the result obtained in Ex. 1 determine the slope of the curve $y = x^2 - 2x + 2$ at the point upon it whose abscissa is $x = 1\frac{1}{2}$. Draw the graph of the curve and indicate in it the geometric significance of your result concerning the slope when $x = 1\frac{1}{2}$. *Ans.* 1.

3. Find the derivative of each of the following functions:

(*a*) $y = 3x^2 - 2$; *Ans.* $y' = 6x$;

(*b*) $y = x^3 - 2x^2 + x$; *Ans.* $y' = 3x^2 - 4x + 1$;

(*c*) $y = x^4$; *Ans.* $y' = 4x^3$;

(*d*) $y = \frac{1}{x^2}$; *Ans.* $y' = -\frac{2}{x^3}$;

(*e*) $y = x + \frac{1}{x}$; *Ans.* $y' = 1 - \frac{1}{x^2}$;

(*f*) $y = \frac{x^2 - 1}{x + 2}$; *Ans.* $y' = \frac{x^2 + 4x + 1}{(x + 2)^2}$.

4. Determine the slope of each of the curves (*a*), (*b*), etc. of Ex. 3 at the special point at which $x = 1$, using for this purpose the expressions already there determined for the derivatives. Draw diagrams indicating the geometric meanings of the various results.

Ans. The slopes are (*a*) 6; (*b*) 0; (*c*) 4; (*d*) -2; (*e*) 0; (*f*) $\frac{2}{3}$.

The student is advised to repeat the exercise, using other values for x than $x = 1$; for example, $x = -1$, $x = \frac{1}{2}$, $x = -\frac{1}{2}$, etc.

5. If $y = \frac{1}{(1 + x)^2}$, show that $D_x y = -\frac{2}{(1 + x)^3}$.

6. If $s = \frac{1 + t + t^2}{t}$, show that $s' = 1 - \frac{1}{t^2}$.

7. Find the coördinates (x, y) of the point P upon the parabola $y = 4x^2 + 8x + 5$ at which the slope is equal to 4. Draw a figure.

Ans. $x = -\frac{1}{2}$, $y = 2$.

8. Show that two tangents can be drawn to the curve $y = 2x^3 + 3x^2 - 12x + 1$ each of which is parallel to the X-axis. Find the point of contact of each.

Ans. $(1, -6)$; $(-2, 21)$.

9. At the point (2, 1) lying upon the curve $8y = x^3$ a tangent is drawn. Find to the nearest 10 minutes the acute angle which it makes with the positive direction of the X-axis. See tables, pages 362–366. Confirm your result by a carefully drawn figure. *Ans.* 56° 20′

10. Find the point upon the curve mentioned in Ex. 9 at which the tangent line makes an angle of 45° with the positive X-axis. *Ans.* $\left(\frac{2\sqrt{6}}{3}, \frac{2\sqrt{6}}{9}\right)$.

11. Prove that there is no point upon the parabola $y = 4x^2$ at which the tangent is parallel to the Y-axis.

12. Find the point, or points, upon the curve $y = 2x^3 - 3x^2 - 6x + 1$ at which the tangent is perpendicular to the line $x + 6y + 1 = 0$.

Ans. $(2, -7)$, $(-1, 2)$.

HINT. Two lines are perpendicular to each other if the slope of the one is the negative reciprocal of the slope of the other.

13. Prove that no curve of the type $y = ax^3 + bx + c$, where a, b, c are any constants except that a and b have the same sign, can have a tangent line parallel to the X-axis.

14. Determine by the method of Art. 9 the derivative of each of the following functions. The letter a, wherever used, represents any constant:

(*a*) $f(x) = 2x^2 + x + 1$; (*d*) $y = \frac{1}{1 + x}$; (*g*) $f(x) = \frac{x^2 + a}{x + a}$;

(*b*) $y = (x - 1)^3$; (*e*) $y = \frac{1}{1 + x^2}$; (*h*) $s = \frac{t^2 + a}{t^2 - a}$;

(*c*) $F(x) = (x + 3)(2 - x)$; (*f*) $y = \frac{1 - t}{1 + t}$; (*i*) $y = \frac{a - x + x^2}{x^3}$.

15. Find the slope of the curve $y = \frac{1}{x}$ at the point $(1, 1)$. Draw a figure.

16. Find the slope of each of the following curves at the point indicated:

(*a*) $y = 5x^2 - 6$ at $(1, -1)$; (*d*) $y = \frac{1}{1 - x}$ at $(0, 1)$;

(*b*) $y = 3x^3 - 5x^2 + 1$ at $(2, 5)$; (*e*) $y = \frac{x + 1}{x - 1}$ at $(2, 3)$;

(*c*) $y = (x^2 + 1)(2 - x)$ at $(0, 2)$; (*f*) $y = \frac{x^2 + 1}{x^2 - 1}$ at $(2, \frac{5}{3})$.

17. Find the points on the curve

$$y = x^3 + x^2 - 4x + 3$$

at which the tangent makes an angle of 45° with the positive X-axis.

10. Increasing and decreasing functions. Inasmuch as the derivative of a function $f(x)$ measures the slope of the curve $y = f(x)$ at each of its points, it follows in particular that if the derivative at a certain point P has a positive value, then the slope of the tangent line at that point is positive. But this requires, by trigonometry, that the angle XQP be acute, as in Fig. 5(*a*), and this in turn implies that the curve is *ascending* at P as one passes along it from left to right. In other words, if the derivative at P is positive, the value of the function y is *increasing* at this point. On the other hand, if the derivative at P is *negative*, then the slope of the tangent line at P is negative. But this requires by trigonometry that the angle XQP be obtuse, as in Fig. 5(*b*), and this implies that the curve is *descending* at P as one passes along it as before from left to right. In other words, the function y is then *decreasing* at this point. These facts may now be summarized as follows:

Theorem. *As the variable x passes by increasing values through any fixed value, as x_1, the function $y = f(x)$ will be increasing or decreasing according as its derivative $f'(x)$, when evaluated at $x = x_1$, is positive or negative in sign; that is, according as $f'(x_1) > 0$ or $f'(x_1) < 0$.*

For example, having found in Example 1, Art. 9, that the derivative of the function $2x^2 - 8x + 7$ is $4x - 8$, it follows that as x passes through the value 3 this function is increasing, for when $x = 3$ the value of $4x - 8$ is 4 and hence is positive. Similarly, we see that as x passes through any fixed value x_1 greater than 2, the same function is increasing, for $4x_1 - 8$ is positive so long as x_1 is greater than 2. On the other hand, as x passes through any fixed value x_1 less than 2, the function is decreasing, for the value of $4x_1 - 8$ is evidently negative whenever x_1 is less than 2. Note how these conclusions are confirmed in Fig. 6.

11. The derivative as a measure of rates. Let us now suppose that $P = (x, y)$ is a point upon the curve $y = f(x)$ at which the derivative is positive in sign so that, by the theorem of Art. 10, the values of y are increasing as we pass along the curve from left to right through P, as illustrated in Fig. 5(*a*). Moreover, let us suppose that in thus passing along the curve we arrive at this point P at a certain time which we will call t. Also, let us suppose that during a small interval of time immediately afterward, which we will call Δt, the continued forward motion along the curve has brought us to the point P', the values of x and y thus having taken on the increments Δx and Δy respectively. Then, upon writing the ratio $\frac{\Delta y}{\Delta x}$ in the form

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}, \tag{1}$$

the numerator $\frac{\Delta y}{\Delta t}$, being the distance Δy divided by the time Δt, is (by elementary mathematics) the average rate of increase of y during the time Δt. Likewise, the denominator $\frac{\Delta x}{\Delta t}$ is the average rate of increase of x during the time Δt. But if Δt be taken very small, these average rates are very nearly equal to the *exact* rates of increase of y and x at P, approaching these exact rates as limits when Δt approaches zero. In this case both Δy and Δx also approach zero, so that, by Art. 9, the ratio $\frac{\Delta y}{\Delta x}$ approaches the value of the derivative

of $f(x)$ at P. If, then, we pass to the limit in (1) as Δt approaches zero, keeping in mind what has just been said of the limits approached by the three expressions $\frac{\Delta y}{\Delta t}$, $\frac{\Delta x}{\Delta t}$, and $\frac{\Delta y}{\Delta x}$ contained in (1), we arrive at the following important theorem:

Theorem I. *If, at the point $P = (x, y)$ upon the curve $y = f(x)$, y increases as x increases* (Fig. 5(*a*)), *then the sign of the derivative $f'(x)$ at P will be* ***positive*** *and its numerical value will be equal to the ratio of the rate of* ***increase*** *of y to the rate of increase of x at this point.*

If, on the other hand, we consider a point P at which y is decreasing as x increases, as in Fig. 5(*b*), then Δy in the foregoing reasoning will be negative in sign. Thus we arrive at the following theorem analogous to Theorem I:

Theorem II. *If, at the point $P = (x, y)$ upon the curve $y = f(x)$, y decreases as x increases* (Fig. 5(*b*)), *then the sign of the derivative $f'(x)$ at P will be* ***negative*** *and its numerical value will be equal to the ratio of the rate of* ***decrease*** *of y to the rate of increase of x at this point.*

We observe finally that the conclusions of both Theorems I and II may be embodied in a single formula. Thus, we have

$$f'(x) = \frac{v_y}{v_x}, \tag{2}$$

where v_x is the rate of change (always regarded as positive in sign) of x at the point $P = (x, y)$ upon the curve $y = f(x)$ and where v_y is the rate of change of y at the same point, it being understood that it is taken positive or negative in sign according as y is increasing or decreasing at P.

Example 1. A point is moving upon the parabola $y = 2x^2 - 8x + 7$ in the general direction from left to right (see Fig. 6). Show

(*a*) that as it crosses the Y-axis its ordinate is *decreasing* eight times as fast as its abscissa is increasing.

(*b*) that as it passes through the point (3, 1) its ordinate is *increasing* four times as fast as its abscissa.

SOLUTION. (*a*) By Example 1, Art. 9, the derivative of $2x^2 - 8x + 7$ is $4x - 8$. Hence, the value of the derivative at the point where the curve cuts the Y-axis, that is, where $x = 0$, is -8. At this point formula (2) thus gives

$$-8 = \frac{v_y}{v_x},$$

from which we obtain

$$v_y = -8\,v_x.$$

Thus, y is changing eight times as fast as x at the point in question, and the fact that v_y has thus been found to be negative indicates that y is decreasing at this point, as is also confirmed by Fig. 6.

(*b*) At the point (3, 1) the value of the derivative is likewise found to be $+4$. Hence, using formula (2), we shall have at this point $\frac{v_y}{v_x} = 4$, or $v_y = 4\,v_x$. Therefore, at the point in question y is increasing four times as fast as x.

Example 2. A bead slides upward along the curve $y = x^3$ (see Fig. 7). If it passes through the point (1, 1) with a velocity of 5 inches per second, how fast is it rising vertically when at this point?

SOLUTION. Since the derivative of x^3, as found in Example 2, Art. 9, is $3\,x^2$, and the value of this expression at the given point (1, 1) is $3\ 1^2 = 3$, it follows from formula (2) that, if v_y and v_x represent the velocities of the bead in the y- and x-directions respectively at this point, we shall have

$$v_y = 3\,v_x. \tag{3}$$

Moreover, since it is given that the bead is moving along the curve with a velocity of 5 inches per second, we also have by a familiar principle in elementary physics the following relation:

$$v_x^2 + v_y^2 = 5^2 = 25.* \tag{4}$$

It remains but to solve (3) and (4) for v_y in order to find the desired vertical velocity, and this may be done by elementary algebra as follows:

From (3) we have $v_x = \frac{1}{3}\,v_y$. Placing this value for v_x in (4), we obtain

$$\frac{v_y^2}{9} + v_y^2 = 25$$

which, when cleared of fractions and solved for v_y^2, reduces to

$$v_y^2 = 22.5.$$

---

* The principle referred to is known as the parallelogram law for the composition of velocities. It states that if the x-component of the velocity of a moving body at P is represented by v_x and its y-component by v_y, then the actual, or resultant velocity v will be in the direction of the diagonal of the parallelogram having v_x and v_y as adjacent sides and the magnitude of v will be equal to the length of this diagonal. In case v_x and v_y are at right angles to each other, as in the present example, each of the two triangles within the parallelogram is a right triangle so that

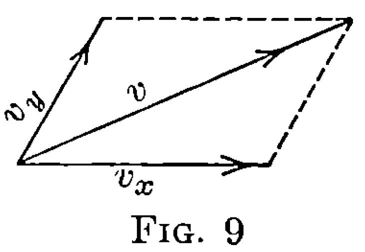

FIG. 9

$$v_x^2 + v_y^2 = v^2.$$

Thus, we have relation (4).

Hence,

$$v_y = \sqrt{22.5} = 4.74^+.$$

The desired vertical velocity of the bead at the point (1, 1) is therefore 4.74 inches per second, approximately.

EXERCISES

Whenever motion along a curve is referred to in the following exercises it is assumed to be in the general direction from left to right, as in the foregoing illustrative examples.

1. Determine by means of the derivative whether the curve

$$y = 3x^2 + 6x + 5$$

is ascending or descending at each of the following points, and confirm your conclusions by means of a diagram containing the curve drawn on coördinate paper:

(*a*) (0, 5); (*b*) (1, 14); (*c*) $(-2, 5)$; (*d*) $(-\frac{1}{3}, 3\frac{1}{3})$;

2. It being given that the derivative of the function $x^3 - 3x^2 - x + 3$ is $3x^2 - 6x - 1$, answer the following questions concerning the curve $y = x^3 - 3x^2 - x + 3$:

(*a*) What is the slope at the point (1, 0)? *Ans.* -4.

(*b*) Is the curve ascending or descending at (1, 0)? *Ans.* Descending.

(*c*) At what rate is y changing at (1, 0) in case x is increasing there at the rate of 2 inches per second? Is this change in y an increase or a decrease?

Ans. Decreasing at 8 inches per second.

(*d*) Confirm, so far as possible, your answers to (*a*), (*b*), (*c*) by drawing the graph of the curve in question and noting its properties.

3. Answer all parts of Ex. 2 for the point $(2, -3)$ instead of (1, 0). Do the same also for each of the following points, all of which are readily found to lie on the given curve since their coördinates in each instance satisfy its equation.

(*a*) (0, 3); (*b*) $(-1, 0)$; (*c*) (3, 0); (*d*) $(-2, -15)$; (*e*) $(\frac{1}{2}, 1\frac{7}{8})$.

4. Show by means of the derivative that the curve

$$y = x^3 - 3x + 1$$

is always ascending except when x lies between -1 and $+1$ and that in this interval it is descending. Confirm these statements by plotting a number of points upon the curve, particularly points having abscissas lying between -2 and $+2$, thus determining geometrically the general form of the curve.

5. Referring again to the curve considered in Ex. 4, answer the following questions: If a point P moves along the curve in such a way that when at (0, 1) the component of its velocity in the direction of the X-axis is 2 inches per second,

(*a*) What is the component of its velocity at this point in the direction of the positive Y-axis?

(*b*) What is its actual velocity at the same point along the curve itself?

(*c*) Answer the same questions also for the points (2, 3), $(-2, -1)$, and $(-1, 3)$.

Ans. (*a*) -6 in. per sec.; (*b*) $2\sqrt{10} = 6.32$ in. per sec.; (*c*) $+18$ in. per sec., 18.11^+ in. per sec.; $+18$ in. per sec., 18.11^+ in. per sec.; 0 in. per sec., 2 in. per sec.

6. If a bead slides down the portion of the curve $y = 1/x$ lying in the first quadrant (Fig. 8), prove that as it passes through the point nearest the origin the vertical and horizontal components of its velocity will be equal, but that this will not be the case at any other point upon the path.

7. Determine for each of the following paths of a moving point, (1) the magnitude of the y-component of the velocity, (2) the magnitude and direction of the resultant velocity, it being given that the point under consideration is the one for which $x = 2$ and that at this point the x-component of the velocity is $v_x = 3$:

(*a*) $y = 2x^2 + x + 1$; (*d*) $y = \dfrac{1}{1-x}$; (*g*) $y = \dfrac{x}{1-x^2}$;

(*b*) $y = (x-2)^3$; (*e*) $y = \dfrac{1}{1-x^2}$; (*h*) $y = \dfrac{3x^2}{1-x^3}$;

(*c*) $y = (x+2)(x-3)$; (*f*) $y = \dfrac{1+x}{1-x}$; (*i*) $y = \dfrac{x^3-1}{x^2}$.

8. A point P moves upward along the parabola $y = 4x^2$ at the constant speed of 2 inches per second. How fast is it rising vertically at the point (1, 4)? How fast is it moving horizontally at the same point?

9. Answer Ex. 8 when the curve $y = 4x^3$ is employed. Draw a diagram representing the geometrical significance of your result.

10. Assuming that the form of a soap bubble is spherical, show that as such a bubble is blown up the rate of increase of its volume at any instant is equal to the rate of increase of the radius multiplied by the area of the sphere at that instant.

11. Show that as a circle is increased (or decreased) in size the rate of increase (or decrease) of the circumference at any instant is directly proportional to the rate of increase (or decrease) of the radius at that instant.

12. Show that as the edge of a cube is increased (or decreased) in size the volume of the corresponding (variable) cube will change at a rate which, at any given instant, will be equal to the rate of change of the edge at that instant multiplied by one half the area of the entire cube at the same instant.

13. A right circular cone whose altitude is always kept equal to the radius of its base is allowed to expand. Show (*a*) that the rate of increase of the volume at any instant will be equal to the rate of increase of the altitude at that instant multiplied by the area of the circular base at the same instant; (*b*) that the rate of increase of the lateral area at any instant will be directly proportional to the circumference of the base at that instant; (*c*) that the rate of increase of the total area (including the base) at any instant will be directly proportional to the altitude at that instant.

14. Investigate all parts of Ex. 14 in case a cone is used whose altitude always remains equal to *twice* the radius of the base.

CHAPTER III

GENERAL THEOREMS CONCERNING THE DERIVATIVE

12. Introduction. The labor required to determine the derivative of a given function may usually be greatly reduced by the use of certain general theorems which we shall state and prove in this chapter, giving illustrations in each instance.

13. The derivative of a constant. The following theorem is first to be noted:

Theorem. *The derivative of any constant is equal to zero.*

When expressed as a formula, this theorem may be written

$$\frac{dc}{dx} = 0,$$

where c represents any constant.

PROOF. Using the notation of Art. 8, we have

$$y = c, \tag{1}$$

and since y in this case remains unchanged no matter what values be assigned to x or to Δx, we must have $\Delta y = 0$. It follows that

$$\frac{\Delta y}{\Delta x} = 0,$$

and hence

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{dc}{dx} = 0,$$

thus establishing the theorem.

When considered geometrically, this theorem states that the graph of equation (1) has its slope equal to zero at every point — a fact which is evident since (1) is the equation of a straight line parallel to the X-axis.

14. The derivative of x^n, n being a positive integer.

Theorem. *The derivative of the function*

$$y = x^n, \tag{1}$$

where n is any positive integer, is given by the formula

$$\frac{dy}{dx} = nx^{n-1}.$$

PROOF. From (1) we have

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^n \\ &= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!} x^{n-2}\overline{\Delta x}^2 + \quad + \overline{\Delta x}^n. \end{aligned} \tag{2}$$

Subtracting (1) from (2), member from member, we obtain

$$\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{2!} x^{n-2}\overline{\Delta x}^2 + \quad + \overline{\Delta x}^n$$

Hence,

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2}\Delta x + \quad + \overline{\Delta x}^{n-1},$$

and therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}.$$

The proof is thus complete.

For example, we have at once such results as the following:

$$\frac{d(x^3)}{dx} = 3\,x^2; \quad \frac{d(x^4)}{dx} = 4\,x^3; \quad \frac{d(x^8)}{dx} = 8\,x^7; \quad \frac{d(x^{10})}{dx} = 10\,x^9; \text{ etc.}$$

Corollary. By taking $n = 1$, the theorem gives in particular the following result:

$$\frac{d(x)}{dx} = 1.$$

Geometrically considered, this equation states the evident fact that the slope of the straight line $y = x$ is equal to 1 at all points.

15. The derivative of a sum; of a product; of a quotient.

Theorem I. *The derivative of the sum of two functions is equal to the sum of their derivatives.*

Expressed as a formula, this theorem may be written

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx},$$

where u and v are any two functions of x.

PROOF. Let

$$y = u + v \tag{1}$$

and suppose that x takes an increment Δx. Denoting by Δu and Δv the increments which u and v then take respectively, we have

$$y + \Delta y = u + \Delta u + v + \Delta v. \tag{2}$$

Subtracting (1) from (2), member from member, we obtain

$$\Delta y = \Delta u + \Delta v.$$

Therefore

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

and hence

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{du}{dx} + \frac{dv}{dx}.$$

The proof is thus complete.

The theorem is seen at once to apply not only when two functions u and v are added, but when any number of functions are either added to or subtracted from one another.

Example. Find $\frac{dy}{dx}$ if

$$y = x^3 + x^2 - x + 5.$$

SOLUTION. Applying Theorem I, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(5).$$

By use of the theorems in Arts. 13, 14, this result takes the final form

$$\frac{dy}{dx} = 3\,x^2 + 2\,x - 1.$$

Theorem II. *The derivative of the product of two functions is equal to the first function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first.*

Expressed as a formula, this theorem may be written

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx},$$

where u and v are any two functions of x.

PROOF. Proceeding as in the proof of Theorem I, we have

$$y = uv,$$

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\Delta v + v\Delta u + \Delta u\Delta v.$$

Hence,

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v,$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x},$$

and therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = u\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0}\Delta u\frac{\Delta v}{\Delta x}$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}.$$

The proof is thus complete.

By considering the special case in which $u = c$, a constant, we obtain the following corollary:

Corollary. *The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.*

Expressed as a formula, this corollary may be written in the form

$$\frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

Theorem II may be at once extended to the case in which three functions are multiplied together instead of two. It then takes the following form:

$$\frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}.$$

Similarly, the theorem may be extended to the product of any number of functions.

Example. Find $\frac{dy}{dx}$ if

$$y = 4(x - 1)(x^2 + 1).$$

SOLUTION. We have

$$\frac{dy}{dx} = 4\frac{d}{dx}[(x - 1)(x^2 + 1)] \qquad \text{Cor. Th. II.}$$

$$= 4\left[(x - 1)\frac{d}{dx}(x^2 + 1) + (x^2 + 1)\frac{d}{dx}(x - 1)\right] \qquad \text{Th. II.}$$

$$= 4[(x - 1)\ 2x + (x^2 + 1)\cdot 1] \qquad \text{Th. I.}$$

$$= 4[3x^2 - 2x + 1].$$

Theorem III. *The derivative of the quotient of two functions is equal to the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, divided by the square of the denominator.*

Expressed as a formula, this theorem may be written

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

where u and v are any two functions of x.

PROOF. Proceeding as in the proof of Theorem I, we have

$$y = \frac{u}{v},$$

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v},$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{(v + \Delta v)v},$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v + \Delta v)v},$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{v \lim\limits_{\Delta x \to 0}\dfrac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0}\dfrac{\Delta v}{\Delta x}}{(v + 0)v}$$

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if

$$y = \frac{x^3}{x^2 - 1}.$$

SOLUTION. We have

$$\frac{dy}{dx} = \frac{(x^2 - 1)\dfrac{d(x^3)}{dx} - x^3\dfrac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2} \qquad \text{Th. III.}$$

$$= \frac{(x^2 - 1)\ 3x^2 - x^3\ (2x)}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 3x^2}{(x^2 - 1)^2}.$$

EXERCISES

Differentiate each of the following functions, making use of the general results in Arts. 13–15. The letters a, b, c represent constants:

1. $y = 3x^2 - 2.$ *Ans.* $y' = 6x.$
2. $y = x^3 + 2x^2 - x.$ *Ans.* $y' = 3x^2 + 4x - 1.$
3. $y = ax^2 + bx + c.$ *Ans.* $y' = 2ax + b.$
4. $s = 10t^3 - 5t^2.$ *Ans.* $s' = 30t^2 - 10t.$
5. $r = 1 + b\theta - c\theta^2.$ *Ans.* $r' = b - 2c\theta.$
6. $y = (x^2 - 1)(x^3 + 1).$ *Ans.* $y' = 5x^4 - 3x^2 + 2x.$
   HINT. Use Th. II, Art. 15.
7. $y = (3x^3 - 1)(2x - 3).$ *Ans.* $y' = 24x^3 - 27x^2 - 2.$
8. $y = (x - 1)(x^2 + 1)(2x^3 - 1).$
   Ans. $y' = 12x^5 - 10x^4 + 8x^3 - 9x^2 + 2x - 1.$
9. $s = 3t^2(t + 2)(t^2 - 1).$ *Ans.* $s' = 15t^4 + 24t^3 - 9t^2 - 12t.$
10. $y = (a - bx)(c - dx).$ *Ans.* $y' = 2bdx + ad + cb.$
11. $y = ax^2 - b(x - c)(x - d).$ *Ans.* $y' = 2(a + b)x - b(c + d).$
12. $y = (x - a)(x - b) - (x - c)(x - d).$ *Ans.* $y' = 4x - (a + b + c + d).$
13. $y = \dfrac{x}{1 - x}.$ *Ans.* $y' = \dfrac{1}{(1 + x)^2}.$
14. $y = \dfrac{x^2 - 1}{1 - 2x}.$ *Ans.* $y' = \dfrac{2(x^2 + x + 1)}{(1 + 2x)^2}.$
15. $y = \dfrac{a - x}{a - x}.$ *Ans.* $y' = -\dfrac{2a}{(a + x)^2}.$
16. $y = \dfrac{x^2 - x}{1 - x^2 + x^3}.$ *Ans.* $y' = \dfrac{-x^4 + 2x^3 - x^2 + 2x - 1}{(1 - x^2 + x^3)^2}.$
17. $y = \dfrac{a^2 - x^2}{a^2 + x^2}.$ *Ans.* $y' = -\dfrac{4a^2x}{(a^2 + x^2)^2}.$
18. $y = \dfrac{x^2 + 1}{x - 1}.$
19. $y = \dfrac{x + 1}{x^2 + 1}.$
20. $y = \dfrac{x}{x^2 + x + 1}.$
21. $y = \dfrac{1}{(x + a)^2}.$
22. $y = \dfrac{x + a}{x + b}.$
23. $y = \dfrac{x^2 + a}{x^2 + b}.$

24. Find the slope of the curve $y = x^3 - 3x^2 + 2x$ at each of the points where it crosses the X-axis. Sketch the curve. *Ans.* 2, -1, 2.

25. Prove that the ordinate at every point of the curve $y = \dfrac{x}{1 + x}$ is increasing. Confirm the fact by sketching the curve.

26. A point moves along the curve $y = x - \dfrac{2}{x}$ and, as it passes through $(1, -1)$, the x-component of its velocity is 2 inches per second. What is the y-component of the velocity at the same point? Also, what is the resultant velocity? *Ans.* 6 in. per sec.; $2\sqrt{10}$ in. per sec.

27. Obtain the derivative of each of the following functions:

(*a*) $y = 5x^2 - 4x + 3;$

(*b*) $y = ax^3 + bx^2 + cx + d;$

(*c*) $s = (t^2 + 1)(t^3 - 1);$

(*d*) $y = \frac{x - 1}{x + 2} + 4x^3;$

(*e*) $f(x) = \frac{x^2 - 1}{x^2 + 2} + (x - 1)(x^2 + 3);$

(*f*) $s = \frac{(t - 1)(t^2 + 1)}{t^3}.$

16. The derivative of a function of a function. It frequently happens that the dependent variable y is a function of the independent variable x through some other variable, as u. For example, this is the case if

$$y = u^2 - 3u \quad \textit{where} \quad u = x^3 + 2x^2 - 1.$$

Here the expression for u occurring in the second equation might be substituted for u in the first equation, whereupon y would become expressed directly as a function of x, after which the derivative y' might be obtained in the usual manner. But to find y' by this roundabout process usually entails considerable work. It is therefore desirable to note the following theorem whereby y' in such cases may be more directly obtained:

Theorem. *If y is a function of u and u is a function of x, thus making y a function of x through u, the derivative y' may be obtained by use of the following formula:*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

PROOF. Suppose $y = f(u)$ where $u = \phi(x)$. Assign to x an increment Δx and let Δu and Δy be the increments resulting in u and y. Then we may write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Hence, allowing Δx to approach zero as a limit, we obtain

$$\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}, \qquad \text{Th. II, Art. 5.}$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \qquad \text{Def., Art., 9.}$$

The proof is thus complete.*

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* The foregoing proof tacitly assumes that at no step in the process do we have $\Delta u = 0$. For the exceptional cases in which this happens the theorem is still true but a more extended proof is necessary.

Example. Find $\dfrac{dy}{dx}$ if

$$y = u^2 - 3\,u \quad \textit{where} \quad u = x^3 + 2\,x^2 - 1. \tag{1}$$

Solution. We have

$$\frac{dy}{dx} = \frac{d}{du}(u^2 - 3\,u)\ \frac{d}{dx}(x^3 + 2\,x^2 - 1)$$
$$= (2\,u - 3)(3\,x^2 + 4\,x). \qquad \text{Ths., Arts. 13–15.}$$

Substituting for u its value as given in the second of equations (1) and making simple reductions, we thus obtain as the final answer

$$\frac{dy}{dx} = (2\,x^3 + 4\,x^2 - 5)(3\,x^2 + 4\,x).$$

17. The derivative of the general power function. As an application of the theorem of Art. 16, we shall now state and prove the following important theorem which may be regarded as a generalization of the theorem of Art. 14.

Theorem. *The derivative of the function*

$$y = u^n, \tag{1}$$

where u is any given function of x and where the exponent n is any rational number (positive or negative, integral or fractional), is expressed by the formula

$$\boldsymbol{\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.}$$

Proof. It will be convenient to consider first the case in which n is positive.

Case 1. *n positive.* Since n is by hypothesis a *rational* number, the function (1) may be regarded as of the form

$$y = u^q, \tag{2}$$

where p and q are positive integers. Hence, we may write

$$y^q = u^p. \tag{3}$$

Differentiating each member of this equation with respect to x, making use of the theorem of Art. 16, we obtain

$$\frac{d}{dy}(y^q)\cdot\frac{dy}{dx} = \frac{d}{du}(u^p)\cdot\frac{du}{dx}$$

which, by the theorem of Art. 14, becomes

$$qy^{q-1}\cdot\frac{dy}{dx} = pu^{p-1}\cdot\frac{du}{dx}.$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{y^{q-1}} \cdot \frac{du}{dx} = \frac{p}{q}\frac{u^{p-1}}{y^{q}} \cdot y \cdot \frac{du}{dx}.$$

or, substituting for y and y^q their values as given in (1) and (2), we have

$$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{u^{p}} \cdot u^{\frac{p}{q}} \cdot \frac{du}{dx} = \frac{p}{q} u^{\frac{p}{q}-1} \frac{du}{dx}.$$

But this result agrees with the formula stated in the theorem when $n = \frac{p}{q}$. The proof is thus complete for Case 1.

Case 2. *n negative.* Equation (1) may now be put in the form

$$y = u^{-m},$$

where m is a positive rational number. This equation may, however, be written as

$$y = \frac{1}{u^m}. \tag{4}$$

Differentiating both members of (4) by the use of Theorem III of Art. 15, we obtain

$$\frac{dy}{dx} = \frac{u^m\frac{d(1)}{dx} - 1 \cdot \frac{d}{dx}u^m}{u^{2m}}.$$

Recalling that $\frac{d(1)}{dx} = 0$ and that by Case I $\frac{d}{dx}u^m = mu^{m-1}\frac{du}{dx}$, we thus obtain

$$\frac{dy}{dx} = \frac{-mu^{m-1}\frac{du}{dx}}{u^{2m}} = -mu^{-m-1}\frac{du}{dx}.$$

But this result agrees with the formula stated in the theorem when $n = -m$. The proof is thus complete for Case II also.

We shall assume hereafter that the theorem holds true also when n is an incommensurable number. The proof for this case lies beyond the scope of this text.

As a special but very important consequence of the theorem obtained from it by taking $u = x$, we have the following corollary:

Corollary. *The derivative of the function*

$$y = x^n,$$

where n is any real number, is given by the formula (compare Art. 14)

$$\frac{dy}{dx} = nx^{n-1}.$$

The following two special formulas which result respectively by placing $n = -1$ and $n = \frac{1}{2}$ in the theorem, are also to be noted:

$$\frac{d\left(\frac{1}{u}\right)}{dx} = -\frac{\frac{du}{dx}}{u^2}, \tag{5}$$

$$\frac{d\sqrt{u}}{dx} = \frac{\frac{du}{dx}}{2\sqrt{u}}. \tag{6}$$

Example 1. Find $\frac{dy}{dx}$ if

$$y = 3(x^2 - 1)^{\frac{1}{4}}.$$

SOLUTION. The theorem gives

$$\frac{dy}{dx} = 3 \quad \frac{1}{4}(x^2 - 1)^{\frac{1}{4}-1} \quad \frac{d}{dx}(x^2 - 1)$$

$$= \frac{3}{4}(x^2 - 1)^{-\frac{3}{4}} \quad 2x = \frac{3x}{2(x^2 - 1)^{\frac{3}{4}}}.$$

Example 2. Find $\frac{dy}{dx}$ if

$$y = \frac{1}{x^2 - x + 1}.$$

SOLUTION. Formula (5) gives

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(x^2 - x + 1)}{(x^2 - x + 1)^2} = -\frac{2x - 1}{(x^2 - x + 1)^2}.$$

Example 3. Find $\frac{dy}{dx}$ if

$$y = \sqrt{ax + b}.$$

SOLUTION. Formula (6) gives

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(ax + b)}{2\sqrt{ax + b}} = \frac{a}{2\sqrt{ax + b}}.$$

EXERCISES

Differentiate each of the following functions. The letters a, b, c represent constants:

1. $y = x^{\frac{2}{3}} + 3\,x^{\frac{1}{3}}$. *Ans.* $y' = \frac{2}{3}x^{-\frac{1}{3}} + x^{-\frac{2}{3}}$.

2. $y = 4\,x^{-\frac{1}{2}} - x^{\frac{1}{2}} + 2$. *Ans.* $y' = -\,2\,x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$.

3. $y = \sqrt{3\,x} + 2\sqrt[3]{x} - \frac{1}{x}$. *Ans.* $y' = \frac{1}{2}\sqrt{\frac{3}{x}} + \frac{2}{3\sqrt[3]{x^2}} + \frac{1}{x^2}$.

4. $y = ax^{-m} + bx^{\frac{n}{2}}$. *Ans.* $y' = -\,max^{-m-1} + \frac{bn}{2}x^{\frac{n-2}{2}}$.

5. $y = \sqrt{x}(\sqrt[3]{x} - 1)$. *Ans.* $y' = \frac{5}{6}x^{-\frac{1}{6}} - \frac{1}{2}x^{-\frac{1}{2}}$.

6. $y = (x+1)\sqrt{x-1}$. *Ans.* $y' = \dfrac{3\,x-1}{2\sqrt{x-1}}$.

7. $y = (x^2+1)\sqrt{x^2-1}$. *Ans.* $y' = \dfrac{3\,x^3 - x}{\sqrt{x^2-1}}$.

8. $f(x) = (a+x)^m(b+x)^n$. *Ans.* $f'(x) = (a+x)^m(b+x)^n\left[\dfrac{m}{a+x} + \dfrac{n}{b+x}\right]$

9. $f(s) = \sqrt{a+bs^2}$. *Ans.* $f'(s) = \dfrac{bs}{\sqrt{a+bs^2}}$.

10. $y = (2\,x^3 - x^2)^{\frac{3}{4}}$. *Ans.* $y' = \frac{3}{2}(3\,x^2 - x)(2\,x^3 - x^2)^{-\frac{1}{4}}$.

11. $s = \sqrt{a + b\sqrt{t}}$. *Ans.* $s' = \dfrac{b}{4\sqrt{at + bt\sqrt{t}}}$.

12. $y = \dfrac{x}{\sqrt{1-x^2}}$. *Ans.* $y' = \dfrac{1}{(1-x^2)^{\frac{3}{2}}}$.

13. $y = \sqrt{\dfrac{x}{x-1}}$. *Ans.* $y' = -\dfrac{1}{2\sqrt{x(x-1)^3}}$.

14. $w = \sqrt{\dfrac{1+z}{1-z}}$. *Ans.* $w' = \dfrac{1}{(1-z)\sqrt{1-z^2}}$.

15. $\phi(t) = \dfrac{\sqrt[3]{t}}{1+\sqrt[3]{t}}$. *Ans.* $\phi'(t) = \dfrac{1}{3\sqrt[3]{t^2}(1+\sqrt[3]{t})^2}$.

16. $y = \sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$. *Ans.* $y' = \dfrac{2\,a^2x}{(a^2-x^2)\sqrt{a^4-x^4}}$.

17. $F(t) = \left(\dfrac{t}{1-t}\right)^m$ *Ans.* $F'(t) = \dfrac{mt^{m-1}}{(1-t)^{m+1}}$.

18. $y = \dfrac{\sqrt{a-x}+\sqrt{a+x}}{\sqrt{a-x}-\sqrt{a+x}}$. *Ans.* $\dfrac{a^2 + a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}$.

19. Find, correct to two significant figures, the slope of the curve $y = \sqrt{x}$ at the point upon it whose abscissa is 2. Draw a figure. *Ans.* 0.35.

20. Find the slope of the circle $x^2 + y^2 = 25$ at the point (3, 4) which lies upon it. *Ans.* $-\frac{3}{4}$.

HINT. Solve the equation for y, being careful to take the sign of the radical adapted to the problem.

21. A point moves outward along the portion of the parabola $y^2 = 8x$ which lies in the first quadrant. If it passes through the point (2, 4) with a velocity of 2 inches per second, how fast is it rising vertically at that point? Compare Example 2, Art. 11. *Ans.* $\sqrt{2}$ in. per sec.

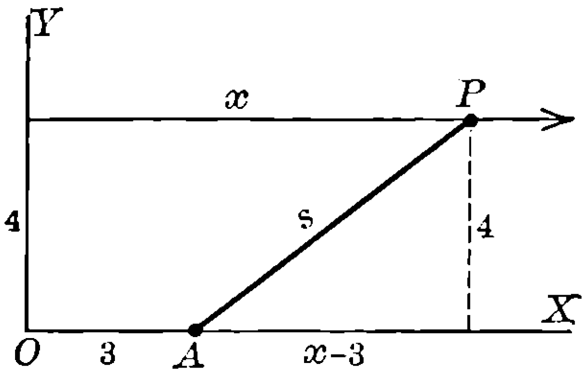

FIG. 10

22. A point P starting at the Y-axis moves to the right upon the line $y = 4$ with a constant velocity of 5 inches per second. Show that at the end of $1\frac{1}{5}$ seconds its distance from the fixed point (3, 0) will be increasing at the rate of 3 inches per second.

HINT. Let x be the variable distance of P from the Y-axis, and let s be its distance from the given point (3, 0). First express s as a function of x.

23. In the adjoining figure the line OL makes the fixed angle θ with the line OX. If a point P moves outward upon OL in the direction of the arrow with a constant velocity v, show that the rate r at which it is receding from the fixed point A situated upon OX at the distance a units from O is given by the formula

$$r = \frac{(x - a\cos\theta)v}{\sqrt{a^2 + x^2 - 2ax\cos\theta}},$$

where x is the (variable) distance of P from O.

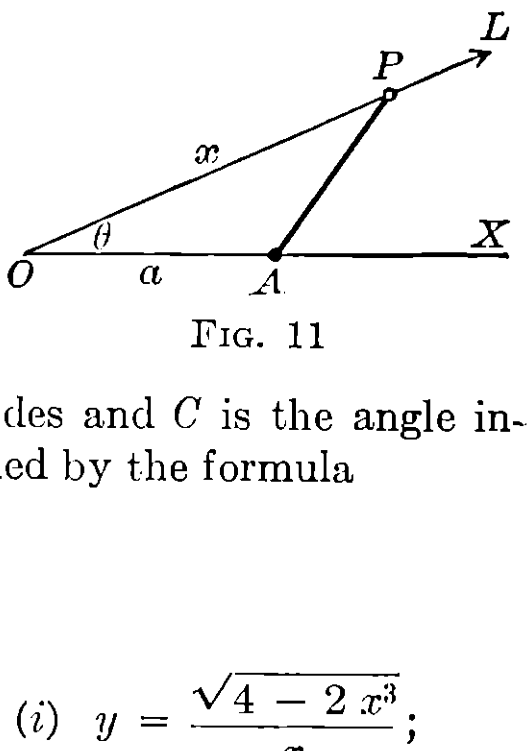

FIG. 11

HINT. Use the law of cosines from trigonometry which states that in any triangle if a and b are two sides and C is the angle included between them, then the third side c is determined by the formula

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

24. Differentiate each of the following functions:

(a) $y = 3x^{\frac{1}{3}} - 2x^{\frac{1}{2}}$; (e) $y = \frac{\sqrt{x}}{1 + x^2}$; (i) $y = \frac{\sqrt{4 - 2x^3}}{x}$;

(b) $y = 3x^{-\frac{1}{3}} - 2x^{-\frac{1}{2}}$; (f) $f(s) = \sqrt[3]{\frac{1 - s}{(1 + s)^2}}$; (j) $y = \frac{x}{\sqrt{4 - 2x^3}}$;

(c) $y = \sqrt[3]{ax^2}$; (g) $F(x) = \frac{a - x}{\sqrt{2ax - x^2}}$; (k) $y = \frac{2x^2 - 1}{x\sqrt{1 + x^2}}$;

(d) $y = \frac{1 + x^2}{\sqrt{x}}$; (h) $y = x^{a+b} - x^{a-b}$; (l) $s = (at^3 + t)^{\frac{5}{3}}$.

25. Show that, as the altitude of a right circular cone increases, the base remaining constant, the lateral area, considered at any instant, will be increasing at a rate which is inversely proportional to the slant height and directly proportional to the altitude at that instant. See Art. 11.

THE DIFFERENTIATION OF THE TRIGONOMETRIC FUNCTIONS *

18. Introduction. The theorems thus far stated enable one to find the derivatives of a large variety of functions, yet there are many important types of functions whose derivatives cannot be readily obtained without further theorems. Of these, the trigonometric functions may be regarded as the simplest. We proceed therefore to consider the derivatives of functions of this type.

19. The derivative of sin *u*. We shall now establish the following theorem:

Theorem. *The derivative of the function sin u, where u is any given function of the angle x measured in radians, is expressed by the formula*

$$\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}.$$

PROOF. Consider first the simple case in which $u = x$; that is, consider the function

$$y = \sin x.$$

Proceeding in the usual manner to find the derivative (see Art. 9), we have

$$y + \Delta y = \sin(x + \Delta x)$$

and hence

$$\Delta y = \sin(x + \Delta x) - \sin x.$$

Moreover, if we apply to this difference the well-known trigonometric formula

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B),$$

the equation for Δy may be written in the form

$$\Delta y = 2 \cos\left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2}.$$

Dividing both members by Δx, we may thus write

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* The student is advised to review at this point the formulas from trigonometry to be found on page 359.

$$\frac{\Delta y}{\Delta x} = \cos\left(x + \frac{\Delta x}{2}\right) \cdot \left(\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right). \tag{1}$$

Now, as Δx approaches zero the first factor here appearing on the right evidently approaches the value $\cos x$ as a limit, while it can be shown (see page 360) that the second factor (which expresses the ratio of the sine of an angle to the angle itself) will approach the limit 1, provided that the angle Δx $\left(\text{or } \frac{\Delta x}{2}\right)$ is measured in radians.*

Therefore, allowing Δx to approach zero in (1), we obtain

$$\frac{dy}{dx} = (\cos x)\quad 1 = \cos x,$$

or

$$\frac{d}{dx}(\sin x) = \cos x. \tag{2}$$

The theorem in question now follows directly. Thus, by the theorem of Art. 16, we have

$$\frac{d}{dx}(\sin u) = \frac{d}{du}(\sin u) \cdot \frac{du}{dx},$$

which by (2) becomes

$$\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if

$$y = \sin (2\,x^2 - 1).$$

Solution. We have

$$y = \sin u \text{ where } u = 2\,x^2 - 1.$$

Hence, applying the theorem, we may write

$$\begin{aligned}\frac{dy}{dx} &= \cos u \cdot \frac{d}{dx}(2\,x^2 - 1)\\ &= \cos(2\,x^2 - 1)\quad 4\,x = 4\,x\cos(2\,x^2 - 1).\end{aligned}$$

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* For the definition of the radian, and its properties, see page 360.

20. The derivative of cos *u*. We may now readily establish the following theorem:

Theorem. *The derivative of the function cos u, where u is any given function of the angle x measured in radians, is expressed by the formula*

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}.$$

PROOF. Let

$$y = \cos u.$$

Since, by trigonometry, $\cos u = \sin(90^\circ - u) = \sin\left(\frac{\pi}{2} - u\right)$, this equation may be written

$$y = \sin\left(\frac{\pi}{2} - u\right).$$

Differentiating, using the theorem of Art. 19, we obtain

$$\frac{dy}{dx} = \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right) = -\cos\left(\frac{\pi}{2} - u\right)\frac{du}{dx}$$

or, recalling that $\cos\left(\frac{\pi}{2} - u\right) = \cos(90^\circ - u) = \sin u$, we have

$$\frac{dy}{dx} = -\sin u \frac{du}{dx}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if $y = \cos\sqrt{3x + 1}$.

SOLUTION. We have

$$y = \cos u \text{ where } u = \sqrt{3x + 1}. \tag{1}$$

Hence, applying the theorem, we obtain

$$\frac{dy}{dx} = -\sin u \frac{d}{dx}\sqrt{3x + 1}$$

which, by formula (6), Art. 17, becomes

$$\frac{dy}{dx} = -\sin u \cdot \frac{\frac{d}{dx}(3x + 1)}{2\sqrt{3x + 1}} = -\sin u \cdot \frac{3}{2\sqrt{3x + 1}}.$$

Substituting for u its value as given in (1) and making other slight changes, the answer takes the final form

$$\frac{dy}{dx} = -\frac{3 \sin\sqrt{3x + 1}}{2\sqrt{3x + 1}}.$$

21. The derivative of tan *u*. We may now readily establish the following theorem:

Theorem. *The derivative of the function tan u, where u is any given function of the angle x measured in radians, is expressed by the formula*

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}.$$

Proof. Let

$$y = \tan u.$$

Then we may write (see page 359)

$$y = \frac{\sin u}{\cos u}.$$

Differentiating by use of Theorem III of Art. 15,

$$\frac{dy}{dx} = \frac{\cos u \dfrac{d}{dx}(\sin u) - \sin u \dfrac{d}{dx}(\cos u)}{\cos^2 u}$$

$$= \frac{(\sin^2 u + \cos^2 u)\dfrac{du}{dx}}{\cos^2 u} = \frac{\dfrac{du}{dx}}{\cos^2 u} = \sec^2 u \frac{du}{dx}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if $y = \tan \frac{x}{x-1}$.

Solution. We have

$$y = \tan u \text{ where } u = \frac{x}{x-1}. \qquad (1)$$

Hence, applying the theorem, we obtain

$$\frac{dy}{dx} = \sec^2 u \cdot \frac{d}{dx}\left(\frac{x}{x-1}\right),$$

which, by Theorem III of Art. 15, becomes

$$\frac{dy}{dx} = \sec^2 u \; \frac{(x-1)\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \sec^2 u \cdot \frac{(x-1)\ 1 - x\ 1}{(x-1)^2}.$$

Substituting for u its value as given in (1) and making other slight reductions and changes, the answer takes the final form

$$\frac{dy}{dx} = -\frac{1}{(x-1)^2}\sec^2 \frac{x}{x-1}.$$

22. The derivative of cot u. We have the following theorem:

Theorem. *The derivative of the function cot u, where u is any given function of the angle x measured in radians, is expressed by the formula*

$$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}.$$

PROOF. Let

$$y = \cot u.$$

Then we may write (see page 359)

$$y = \frac{1}{\tan u}.$$

Differentiating by use of formula (5) of Art. 17, we obtain

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\tan u)}{\tan^2 u}$$

which, by use of the theorem of Art. 21 and familiar trigonometric formulas (see page 359), becomes

$$\frac{dy}{dx} = -\frac{\sec^2 u \frac{du}{dx}}{\tan^2 u} = -\csc^2 u \frac{du}{dx}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if $y = \cot \frac{1}{x^2}.$

SOLUTION. We have

$$y = \cot u \text{ where } u = \frac{1}{x^2} = x^{-2}. \tag{1}$$

Hence, applying the theorem, we obtain

$$\frac{dy}{dx} = -\csc^2 u \cdot \frac{d}{dx}(x^{-2}),$$

which, by the corollary in Art. 17, becomes

$$\frac{dy}{dx} = -\csc^2 u \cdot (-2x^{-3}).$$

Substituting for u its value as given in (1) and making other slight changes, the answer takes the final form

$$\frac{dy}{dx} = \frac{2}{x^3}\csc^2 \frac{1}{x^2}.$$

23. The derivative of sec *u* and of csc *u*. We have the following theorem:

Theorem. *The derivatives of the functions sec u and csc u, where u is any given function of the angle x measured in radians, are expressed respectively by the formulas*

$$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}, \quad \frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}.$$

PROOF. Let

$$y = \sec u.$$

Then, by trigonometry,

$$y = \frac{1}{\cos u}.$$

Differentiating by use of formula (5) of Art. 17, we obtain

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\cos u)}{\cos^2 u},$$

which, by using theorem of Art. 20 and making simple trigonometric changes, becomes

$$\frac{dy}{dx} = \frac{\sin u \frac{du}{dx}}{\cos^2 u} = \sec u \quad \tan u \quad \frac{du}{dx},$$

thus completing the proof of the first part of the theorem.

In order to prove the second part, let

$$y = \csc u.$$

Then, by trigonometry,

$$y = \frac{1}{\sin u}.$$

Differentiating by use of formula (5) of Art. 17, we obtain

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\sin u)}{\sin^2 u},$$

which, by using the theorem of Art. 19 and making simple trigonometric changes, becomes

$$\frac{dy}{dx} = -\frac{\cos u \frac{du}{dx}}{\sin^2 u} = -\csc u \cdot \cot u \cdot \frac{du}{dx}.$$

The proof of the theorem is thus complete.

Example. Find $\frac{dy}{dx}$ if

$$y = \sec(ax + b).$$

SOLUTION. We have

$$y = \sec u \quad \text{where} \quad u = ax + b.$$

Hence, applying the first part of the theorem,

$$\frac{dy}{dx} = \sec u \tan u \frac{d}{dx}(ax + b),$$

which takes the final form

$$\frac{dy}{dx} = a \sec(ax + b) \tan(ax + b).$$

Note. The student will observe that the unit of angular measurement used in the various formulas of Arts. 19–23 is the radian, this being required in the proof of the formula in Art. 19 upon which the proofs of all the other formulas depend. For this reason the radian is taken as the unit of angular measurement in all higher mathematics. If the ordinary unit, namely the degree, were used, it is not difficult to show that the effect upon the foregoing formulas would be to multiply the second member of each by the constant factor $\frac{\pi}{180}$, which is the ratio of the size of a degree to that of a radian.

EXERCISES

Differentiate each of the following functions. The letters a, b, c represent constants:

1. $y = \sin 4x$. *Ans.* $y' = 4 \cos 4x$.

2. $y = \cos\sqrt{1-x}$. *Ans.* $y' = \frac{\sin\sqrt{1-x}}{2\sqrt{1-x}}$.

3. $y = \tan\sqrt{1-x}$. *Ans.* $y' = -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$.

4. $y = \cot\frac{1}{x^2}$. *Ans.* $y' = \frac{2}{x^3}\csc^2\frac{1}{x^2}$.

5. $y = x^2 + \sin x^2$. *Ans.* $y' = 2x(1 + \cos x^2)$.

6. $y = x^2 \sin x^2$. *Ans.* $y' = 2x(\sin x^2 + x^2 \cos x^2)$.

7. $y = x \sin x + \cos x$. *Ans.* $y' = x \cos x$.

8. $y = \frac{\tan x - 1}{\sec x}$. *Ans.* $y' = \sin x + \cos x$.

9. $f(t) = \frac{\sin t}{t}$. *Ans.* $f'(t) = \frac{t \cos t - \sin t}{t^2}$.

10. $y = \sin^2 x$. *Ans.* $y' = 2 \sin x \cos x = \sin 2x$.

HINT. $\sin^2 x = (\sin x)^2$. See Art. 17.

11. $y = \cos^3 x$. *Ans.* $y' = -3\cos^2 x \sin x$.

12. $y = \tan^2 \sqrt{x}.$ *Ans.* $y' = \dfrac{\tan \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}.$

13. $s = \sqrt{1 + \sin t}.$ *Ans.* $s' = \dfrac{\cos t}{2\sqrt{1 + \sin t}}.$

14. $r = a \sin^3 \dfrac{\theta}{3}.$ *Ans.* $= a \sin^2 \dfrac{\theta}{3} \cos \dfrac{\theta}{3}.$

15. $y = \sin^3 x \cos x.$ *Ans.* $y' = \sin^2 x(3 \cos^2 x - \sin^2 x).$

16. $\rho = \dfrac{1}{1 - \cos \theta}.$ *Ans.* $\dfrac{d\rho}{d\theta} = -\dfrac{\sin \theta}{(1 - \cos \theta)^2}.$

17. $\rho = a\sqrt{\cos 2\theta}.$ *Ans.* $\rho' = -\dfrac{a \sin 2\theta}{\sqrt{\cos 2\theta}}.$

18. $y = \cos \sqrt{x^2 - a^2}.$ *Ans.* $y' = -\dfrac{x \sin \sqrt{x^2 - a^2}}{\sqrt{x^2 - a^2}}.$

19. $f(\theta) = \sin(\cos \theta).$ *Ans.* $f'(\theta) = -\sin \theta \cos(\cos \theta).$

20. $f(x) = \sin(x + a)\cos(x - a).$ *Ans.* $f'(x) = \cos 2x.$

21. From the relation $\sin 2x = 2 \sin x \cos x$, deduce by differentiation the fact that

$$\cos 2x = \cos^2 x - \sin^2 x.$$

22. Prove that the curve $y = \sin x$ (see page 370) meets the X-axis at an angle of 45° at each of the points $x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi,$ and at an angle of 135° at each of the points $x = \pm\pi, \pm 3\pi, \pm 5\pi,$

23. Prove that each branch of the curve $y = \tan x$ (see page 370) meets the X-axis at an angle of 45°

24. Show that $\tan x$ is increasing twice as fast as x itself when $x = \dfrac{\pi}{4}.$ Determine any other positive value of x less than 2π for which the same statement is true.

25. The adjoining figure represents a right triangle whose altitude $CB = 10$ inches but whose side $CA = s$ and whose hypotenuse $AB = h$ are regarded as variables depending upon the size of the angle x. If x increases at the constant rate of one fourth of a radian per second, (a) how fast is s increasing when $x = 60°$? (b) how fast is h increasing when $x = 45°$?

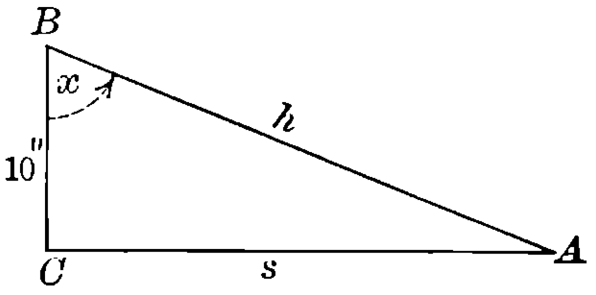


FIG. 12

26. Answer both parts of Ex. 25 when the rate at which the angle x is increasing is 30° per second, other conditions remaining the same.

27. Find the derivative of each of the following functions:

(a) $y = \sin 2x + \sin^2 x$; (d) $y = \tan^2\sqrt{a^2 - x^2}$; ($g$) $y = \dfrac{\sin x}{\cos 3x}$;

(b) $y = \sin 2x \sin^2 x$; (e) $y = \dfrac{\sin^2 x}{1 - x}$; ($h$) $y = \dfrac{\sin x}{\cos^3 x}$;

(c) $y = \tan \sqrt{x^2 - a^2}$; ($f$) $y = \dfrac{\cos^2 x}{1 + x}$; ($i$) $y = x^2 \tan \dfrac{x}{2}.$

THE DIFFERENTIATION OF THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS*

24. The derivative of the function $\log_a u$. We shall now establish the following theorem:

Theorem. *The derivative of the function $\log_a u$, where u is a given function of x, is expressed by the formula*

$$\frac{d}{dx}(\log_a u) = \log_a e \cdot \frac{\frac{du}{dx}}{u},$$

where e is a constant whose value, correct to five decimal places, is 2.71828.

PROOF. Consider first the simple case in which $u = x$; that is, consider the function

$$y = \log_a x.$$

Proceeding in the usual manner to find the derivative (see Art. 9), we have

$$y + \Delta y = \log_a (x + \Delta x)$$

and hence

$$\Delta y = \log_a (x + \Delta x) - \log_a x,$$

or

$$\Delta y = \log_a \left(\frac{x + \Delta x}{x}\right) = \log_a \left(1 + \frac{\Delta x}{x}\right).$$

Therefore,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right),$$

or

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right) = \frac{1}{x} \cdot \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \quad (1)$$

Now, as Δx approaches zero, the factor $\frac{1}{x}$ here appearing on the right evidently remains unchanged. As to the factor $\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$, it can be shown that as Δx approaches zero this expression approaches one and

* The student is advised to review at this point the definitions and general properties of logarithms. See, for example, the author's *College Algebra*, Chap. VII.

the same limit no matter what value x may have; that is, the limit approached is a constant. This constant plays an important rôle in the calculus and all higher mathematics, as will appear in the following chapters. It is usually represented by the letter e and its value, correct to five decimal places, is 2.71828, as will appear in Art. 89.

Therefore, allowing Δx to approach zero in (1), we obtain

$$\frac{dy}{dx} = \frac{1}{x} \log_a e,$$

or

$$\frac{d}{dx}(\log_a x) = \log_a e \cdot \frac{1}{x}. \tag{2}$$

The theorem in question now follows directly. Thus, by the theorem of Art. 16 we have

$$\frac{d}{dx}(\log_a u) = \frac{d}{du}(\log_a u) \cdot \frac{du}{dx},$$

which by (2) becomes

$$\frac{d}{dx}(\log_a u) = \log_a e \cdot \frac{\frac{du}{dx}}{u}. \tag{3}$$

The proof is thus complete.

Remarks. If in particular we select $a = e$, that is, if we take the number e as the base of the system of logarithms used, we shall have $\log_a e = \log_e e = 1$, so that formula (3) then assumes the simple form

$$\frac{d}{dx}(\log u) = \frac{\frac{du}{dx}}{u}, \tag{4}$$

it being understood that the symbol *log* here signifies "logarithm with respect to the base e." On account of the simplicity thus resulting, it is customary to use the base e universally in the calculus. Therefore, formula (4) is hereafter to be used in all cases where the derivative of the logarithm of a function is to be determined and for simplicity $\log_e u$ will hereafter be written $\log u$.

Example. Find $\frac{dy}{dx}$ if $y = \log(3x^2 + 1)$.

SOLUTION. We have $y = \log u$ where $u = 3x^2 + 1$.
Hence, applying formula (4),

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(3x^2 + 1)}{u} = \frac{6x}{3x^2 + 1}.$$

Note. The use of the number e as the base of a system of logarithms originated with Sir John Napier (1550–1617). Such logarithms are known as Naperian or *natural* logarithms and their use antedates that of the common, or Briggs, logarithms which employ 10 as a base. Tables of natural as well as common logarithms are available,* but it may be shown that the natural logarithm of any number results by merely dividing its common logarithm by 0.434294 (approximately). This special number is the value of $\log_{10} e$ and is known as the *modulus* of the system of common logarithms.

25. Functions of the type a^u. We proceed to consider the derivative of any function of the type a^u, where a is any positive constant and where u is a given function of x, functions of this type being known as *exponential functions.* In order to do this, it is first desirable to establish the following general lemma:

Lemma. *The derivative $\frac{dy}{dx}$ of any function*

$$y = f(x) \tag{1}$$

may be expressed in the form

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$

where $\frac{dx}{dy}$ is the derivative with regard to y of the function

$$x = \phi(y) \tag{2}$$

which one obtains by solving the given equation (1) *for* x.

PROOF. By identity we may write

$$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}.$$

Allowing Δx and hence Δy to approach zero in this identity, we obtain by the definition of a derivative in Art. 9, the equation

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

The proof is thus complete.

---

* See for example *Peirce's Integral Tables* (Ginn and Co.), pages 130–138.

For example, in finding the derivative of the function

$$y = (x + 2)^{\frac{1}{3}} \qquad (1)$$

we may proceed as follows:

By cubing both members of (1) and rearranging the result, we may readily solve (1) for x, thus obtaining

$$x = y^3 - 2. \qquad (2)$$

Applying the lemma, we now obtain

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3\,y^2}.$$

This is the desired result, but as here appearing it is expressed in terms of the variable y instead of x. If it is desired to express it in terms of x, we may substitute for y its value as given in (1), thus giving

$$\frac{dy}{dx} = \frac{1}{3(x + 2)^{\frac{2}{3}}}.$$

The student should observe that this final result is the same as that obtained by applying the theorem of Art. 17 directly to the function (1).

Returning now to the function a^u, we proceed to establish the following general theorem:

Theorem. *The derivative of the function a^u, where a is any positive constant and where u is a given function of x, is expressed by the formula*

$$\frac{d}{dx}(a^u) = \log a \cdot a^u \cdot \frac{du}{dx}, \qquad (1)$$

where log signifies "logarithm with respect to the base e."

PROOF. Consider first the simple case in which $u = x$; that is, consider the function

$$y = a^x. \qquad (2)$$

Taking the logarithm to the base e of each side of this equation, we obtain

$$\log y = x \log a,$$

or, solving for x,

$$x = \frac{\log y}{\log a} = \frac{1}{\log a} \cdot \log y. \qquad (3)$$

Having thus solved (2) for x, we may at once apply the foregoing

lemma to determine $\frac{dy}{dx}$. Thus, from (3) we have, by the theorem of Art. 24,

$$\frac{dx}{dy} = \frac{1}{\log a} \cdot \frac{1}{y},$$

so that the lemma gives

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{\log a} \cdot \frac{1}{y}} = \log a \cdot y.$$

Replacing y by its value as given in (2), we obtain

$$\frac{dy}{dx} = \log a \cdot a^x,$$

or

$$\frac{d}{dx}(a^x) = \log a \cdot a^x. \tag{4}$$

The theorem as stated now follows directly. Thus, by the theorem of Art. 16 we have

$$\frac{d}{dx}(a^u) = \frac{d}{du}(a^u) \cdot \frac{du}{dx},$$

which, by (4), becomes

$$\frac{d}{dx}(a^u) = \log a \cdot a^u \cdot \frac{du}{dx}.$$

The proof is thus complete.

Remarks. If in particular $a = e$, formula (1) assumes the following simple but important form:

$$\boldsymbol{\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}}. \tag{5}$$

Example. Find $\frac{dy}{dx}$ if

$$y = e^{\sin x}.$$

SOLUTION. We have

$$y = e^u \quad \text{where} \quad u = \sin x.$$

Hence, applying formula (5),

$$\frac{dy}{dx} = e^u \frac{d}{dx} \sin x = e^{\sin x} \cdot \cos x. \qquad \text{Th., Art. 19}$$

26. Functions of the type u^v. While it is possible to obtain a general theorem concerning the derivative of any function of the type u^v, where u and v are given functions of x, it is usually best to consider each case separately, employing the theorems of Arts. 24, 25. The following example illustrates the usual process employed.

Example. Find $\frac{dy}{dx}$ if

$$y = x^{\sin x}.$$

SOLUTION. Taking the logarithm (base e) of each member of the equation gives

$$\log y = \sin x \cdot \log x.$$

Differentiating this equation member by member, making use of the theorems of Arts. 15, 24, we obtain

$$\frac{\frac{dy}{dx}}{y} = \sin x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(\sin x)$$
$$= \frac{\sin x}{x} + \log x \cdot \cos x.$$

Replacing y by its value as given in terms of x and clearing of fractions, the answer takes the form

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cdot \cos x\right).$$

EXERCISES

Differentiate each of the following functions, it being understood that all logarithms are to the base e unless otherwise indicated. The letters a, b, c represent constants.

1. $y = \log(1 - x^2)$. *Ans.* $\frac{dy}{dx} = -\frac{2x}{1 - x^2}$.

2. $y = \log(ax^2 + bx + c)$. *Ans.* $\frac{dy}{dx} = \frac{2ax + b}{ax^2 + bx + c}$.

3. $y = \log\frac{1 + x}{1 - x}$. *Ans.* $\frac{dy}{dx} = \frac{2}{1 - x^2}$.

4. $y = \log\frac{x}{x^2 - 1}$. *Ans.* $y' = -\frac{x^2 + 1}{x(x^2 - 1)}$.

5. $y = \log_{10}(x^3 + 2x^2)$. *Ans.* $y' = \log_{10} e \cdot \frac{3x + 4}{x^2 + 2x}$.

6. $y = x \log x$. *Ans.* $y' = 1 + \log x$.

7. $y = \log x^3$. *Ans.* $y' = \frac{3}{x}$.

8. $y = (\log x)^3$. *Ans.* $y' = \frac{3(\log x)^2}{x}$.

9. $f(x) = \log(x + \sqrt{1 + x^2})$. *Ans.* $f'(x) = \frac{1}{\sqrt{1 + x^2}}$.

10. $y = \frac{\log(ax + b)}{ax + b}$. *Ans.* $y' = \frac{a[1 - \log(ax + b)]}{(ax + b)^2}$.

11. $y = \log\sqrt{1 + x}$. *Ans.* $y' = \frac{1}{2(1 + x)}$.

HINT. $\log\sqrt{1 + x} = \frac{1}{2}\log(1 + x)$.

12. $y = \log\sqrt{\frac{1 - x}{1 + x}}$. *Ans.* $y' = -\frac{1}{1 - x^2}$.

HINT. $\log\sqrt{\frac{1 - x}{1 + x}} = \frac{1}{2}\log\frac{1 - x}{1 + x}$

$= \frac{1}{2}\log(1 - x) - \frac{1}{2}\log(1 + x)$.

13. $y = \log\sqrt[3]{1 + x^2}$. *Ans.* $y' = \frac{2x}{3(1 + x^2)}$.

14. $y = \log\sqrt{\frac{1 + x^2}{1 - x^2}}$. *Ans.* $y' = \frac{2x}{1 - x^4}$.

15. $y = \log(\tan x)$. *Ans.* $y' = \frac{2}{\sin 2x}$.

16. $y = \tan(\log x)$. *Ans.* $y' = \frac{\sec^2(\log x)}{x}$.

17. $f(x) = \log\sqrt{\cos x}$. *Ans.* $f'(x) = -\frac{1}{2}\tan x$

18. $y = \log\sqrt{\frac{1 + \sin x}{1 - \sin x}}$. *Ans.* $\frac{dy}{dx} = \frac{1}{\cos x}$.

19. $y = e^{ax+b}$. *Ans.* $y' = ae^{ax+b}$.

20. $y = e^{ax^2+bx+c}$. *Ans.* $y' = (2ax + b)e^{ax^2+bx+c}$.

21. $y = e^{\frac{1}{x}}$. *Ans.* $y' = -\frac{1}{x^2}e^{\frac{1}{x}}$.

22. $y = 3^{3x^3}$. *Ans.* $y' = 9x^2 \cdot 3^{3x^3}\log 3$.

23. $y = e^{\log z}$. *Ans.* $y' = 1$.

HINT. $e^{\log z} = z$ from the definition of a logarithm.

24. $y = a^{\log_a z^2}$. *Ans.* $y' = 2z$.

25. $f(x) = e^{x\log x}$. *Ans.* $f'(x) = e^{x\log x}[1 + \log x]$.

26. $w = e^{\tan ax}$. *Ans.* $w' = ae^{\tan ax}\sec^2 ax$.

27. $\rho = e^{\cos\theta}\sin\theta$. *Ans.* $\frac{d\rho}{d\theta} = e^{\cos\theta}[\cos\theta - \sin^2\theta]$.

28. $q = e^r\log\sin r$. *Ans.* $\frac{dq}{dr} = e^r[\cot r + \log\sin r]$.

29. $y = x^x$. HINT. See Art. 26. *Ans.* $y' = x^x(\log x + 1)$.

30. $y = (\sin x)^x$. *Ans.* $y' = (\sin x)^x [\log \sin x + x \cot x]$.

31. $y = (\sin x)^{\sin x}$. *Ans.* $y' = y \cos x [\log \sin x + 1]$.

32. $y = (\sin x)^{\log x}$. *Ans.* $y' = y\left[\cot x \log x + \frac{1}{x} \log \sin x\right]$.

33. $y = (\log x)^{\log x}$. *Ans.* $y' = \frac{y}{x}\left[1 + \log(\log x)\right]$.

34. $y = (\log x)^{\sin x}$. *Ans.* $y' = y\left[\cos x \log(\log x) + \frac{\sin x}{x \log x}\right]$.

35. $y = e^{x \sin x}$. *Ans.* $y' = y[x \cos x + \sin x]$.

36. $y = (\sin x)^{e^x}$ *Ans.* $y' = ye^x [\log \sin x + \cot x]$.

37. $y = e^{x \log x}$. *Ans.* $y' = y[1 + \log x]$.

38. $y = (\log x)^{e^x}$. *Ans.* $y' = ye^x\left[\frac{1}{x \log x} + \log(\log x)\right]$.

39. Show that the curve $y = e^x$ (see page 370) intersects the Y-axis at an angle of 45°.

40. Show that the curve $y = \log x$ (see page 370) intersects the X-axis at an angle of 45°.

41. Show that as a positive number grows larger and larger its logarithm does the same, but that the rate of increase of the logarithm becomes slower and slower as compared to the rate of increase of the number itself. Is this true whatever the base of the system of logarithms?

42. Show that if a point P moves in any continuous manner along the curve $y = e^x$, the ratio of the vertical component of its velocity at any given point P_1 to its horizontal component at that point is measured graphically by the ordinate of P_1.

43. A point P moves in a continuous manner along the curve $y = \log x$. Describe how one can determine graphically the ratio of the vertical component of its velocity to the horizontal component at any fixed point P_1 upon the path. Compare Ex. 42.

44. Find the derivative of each of the following functions:

(*a*) $y = \log(1 + x^2)$; (*f*) $y = \log \cos^2 x$; (*k*) $y = (\cos x)^x$;

(*b*) $y = \log(1 + x)^2$; (*g*) $y = \cos(\log x)$; (*l*) $y = (\cos x)^{\cos x}$;

(*c*) $y = \log \frac{1 + x^2}{1 - x^2}$; (*h*) $y = e^{x \tan x}$; (*m*) $y = x^{\cos x}$;

(*d*) $y = \frac{\log(1 + x^2)}{\log(1 - x^2)}$; (*i*) $y = e^{\tan^2 x}$; (*n*) $y = (\log x)^{x^2}$;

(*e*) $y = \log \cos x^2$; (*j*) $y = a^{\sin^2 bx}$; (*o*) $y = (\log \log x)^2$.

THE DIFFERENTIATION OF THE INVERSE TRIGONOMETRIC FUNCTIONS

27. Definitions. The expression arc sin x (or $\sin^{-1} x$) is used to represent an angle whose sine is x. Such an angle has various determinations; that is, the value of arc sin x for a given value of x is not unique.

For example, arc sin $\frac{1}{2}$, being by definition an angle whose sine is $\frac{1}{2}$, may be taken as 30°, but by trigonometry it may also be taken as any one of the following:

$$150°,\ 390°,\ 510°,\ 750°,$$

also, it may have any one of the following negative values:

$$-210°,\ -330°,\ -570°,\ -690°,$$

Considered in their totality, these possible values of arc sin $\frac{1}{2}$ may be represented as $30° \pm n\,360°$ and $150° \pm n\,360°$, where n is understood to be zero or any positive integer. When expressed in radian measure, these possible values of arc sin $\frac{1}{2}$ become

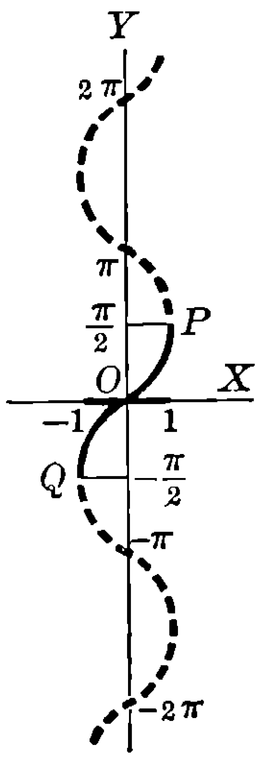
$$\frac{\pi}{6} \pm 2\,n\pi \text{ and } \frac{5\,\pi}{6} \pm 2\,n\pi.$$

Similarly, the possible values of arc sin 1 are $90° \pm n\,360°$, or $\frac{\pi}{2} \pm 2\,n\pi$, while the possible values of arc sin (-1) are $270° \pm 360°$, or $\frac{3\,\pi}{2} \pm 2\,n\pi$.

Keeping in mind this multiple-valued property of arc sin x and noting also that x can take only values lying between -1 and $+1$ (since all sines are thus confined), it readily appears that the graph of the equation

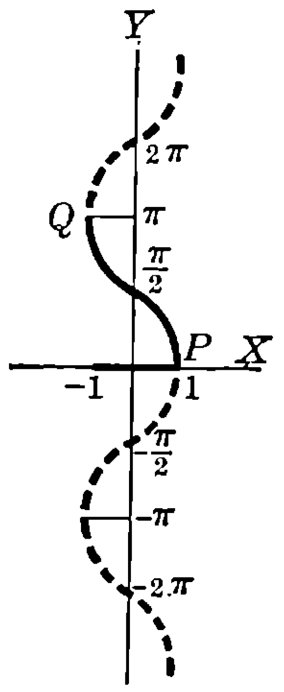
$$y = \text{arc sin } x,$$

where arc sin x is now regarded as a function of x, has the form of the indefinitely long wavy curve shown in Fig. 13. It is important to remark, however, that for the purposes of the calculus it suffices to employ only that part PQ of the curve which lies between $y = -\pi/2$ and $y = +\pi/2$, this part being known as the **principal branch** of the curve. With this restriction, which will be adopted hereafter unless otherwise stated, there is but one value of y corresponding to any given value of x, thus rendering the function arc sin x single-valued instead of multiple-valued.



y = arc sin x

FIG. 13

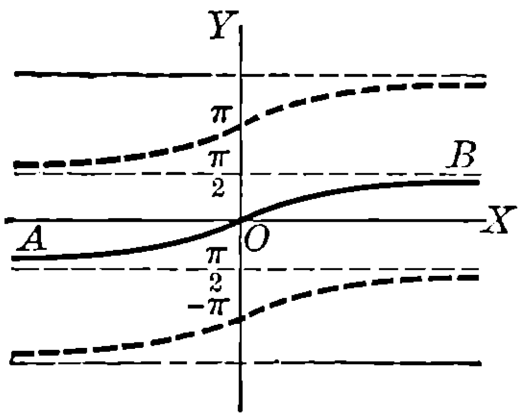


$y = \text{arc} \cos x$
Fig. 14

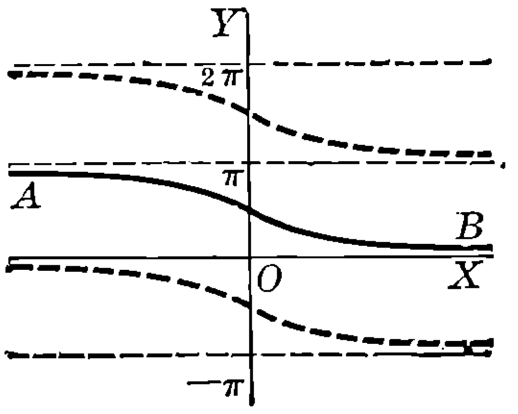
Likewise, the expression arc $\cos x$ (or $\cos^{-1} x$) is used to represent the angle whose cosine is x. It is a multiple-valued function having an unlimited number of values for any one value of x, x itself being necessarily confined within the limits -1 to $+1$ (inclusive).

The graph of this function is shown in Fig. 14, the portion PQ lying between $y = 0$ and $y = \pi$ (principal branch) being the part ordinarily adopted in determining the value of the function y corresponding to a given value of x.

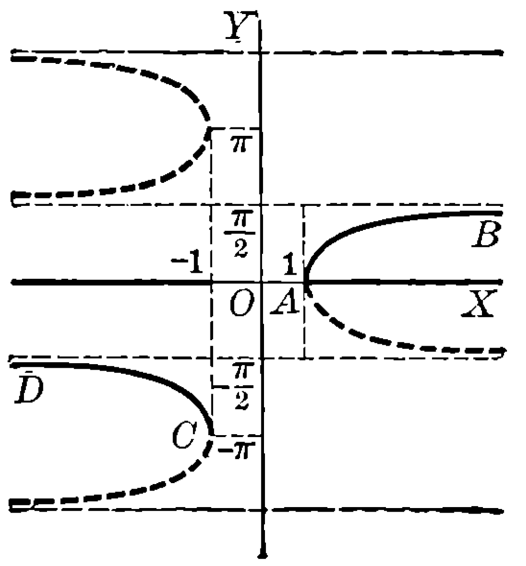
The definitions of arc $\tan x$, arc $\cot x$, arc $\sec x$, and arc $\csc x$ are readily supplied from analogy with the preceding. Each is a multiple-valued function of x having an unlimited number of values for a given value of x. The graph of each consists of an unlimited number of distinct branches, above and below the x-axis, indefinitely long, as indicated in Figs. 15–18. These figures also indicate in each instance



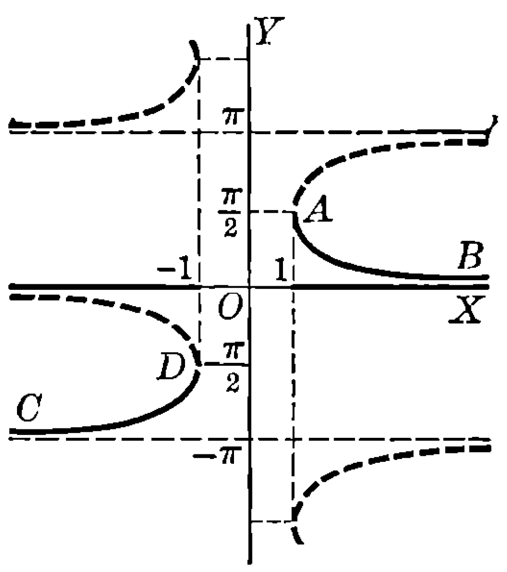
$y = \text{arc} \tan x$
Fig. 15



$y = \text{arc} \cot x$
Fig. 16



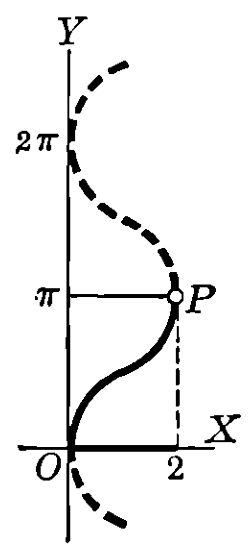
$y = \text{arc} \sec x$
Fig. 17



$y = \text{arc} \csc x$
Fig. 18

the part of the curve (principal branch) ordinarily adopted for determining the value of y corresponding to a given value of x.

Mention is also to be made of the function arc vers x (or $\text{vers}^{-1} x$) which is defined as the angle whose versed sine is x; that is, the angle y such that $1 - \cos y = x$. The graph of this function, with indication of the portion (principal branch) ordinarily adopted in determining y, is shown in Fig. 19.



$y = \text{arc vers } x$
Fig. 19

28. The derivative of arc sin u. We proceed to establish the following general theorem:

Theorem. *The derivative of the function arc sin u, where u is a given function of x, is expressed by the formula*

$$\frac{d}{dx}(\text{arc sin } u) = \frac{\dfrac{du}{dx}}{\sqrt{1 - u^2}}.$$

Proof. Consider first the simple case in which $u = x$; that is, consider the function

$$y = \text{arc sin } x. \tag{1}$$

Then, by the definition of arc sin x in Art. 27, we have

$$x = \sin y \tag{2}$$

and hence, by Art. 19,

$$\frac{dx}{dy} = \cos y.$$

It follows by the lemma of Art. 25 that

$$\frac{dy}{dx} = \frac{1}{\cos y}. \tag{3}$$

But from (2) together with formula (2) of page 359, we may write

$$\cos y = +\sqrt{1 - \sin^2 y} = +\sqrt{1 - x^2} = \sqrt{1 - x^2},$$

where the positive sign is used with the radical because, with y restricted to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (see Fig. 13) the value of $\cos y$ is necessarily positive. Thus, (3) takes the form

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}. \tag{4}$$

The theorem as stated now follows directly. Thus, by Art. 16 we have

$$\frac{d}{dx}(\text{arc} \sin u) = \frac{d}{du}(\text{arc} \sin u) \cdot \frac{du}{dx},$$

which, by (4), becomes

$$\frac{d}{dx} \text{arc} \sin u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if

$$y = \text{arc} \sin \sqrt{x}.$$

SOLUTION. We have

$$y = \text{arc} \sin u \quad \text{where} \quad u = \sqrt{x}.$$

Hence, applying the theorem,

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}} = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}.$$

29. The derivative of arc cos u.

Theorem. *The derivative of the function arc cos u, where u is any given function of x, is expressed by the formula*

$$\frac{d}{dx}(\text{arc} \cos u) = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

PROOF. Consider first the case in which $u = x$; that is, consider

$$y = \text{arc} \cos x. \tag{1}$$

Then, by the definition of arc cos x in Art. 27, we have

$$x = \cos y, \tag{2}$$

and hence, by Art. 20,

$$\frac{dx}{dy} = -\sin y.$$

It follows by the lemma of Art. 25 that

$$\frac{dy}{dx} = -\frac{1}{\sin y}. \tag{3}$$

But from (2) together with formula (2) of page 359 we may write

$$\sin y = +\sqrt{1 - \cos^2 y} = +\sqrt{1 - x^2} = \sqrt{1 - x^2},$$

where the positive sign is used with the radical because, with y restricted to lie between 0 and π (see Fig. 14), the value of $\sin y$ is necessarily positive. Thus, (3) takes the form

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}. \tag{4}$$

The theorem as stated now follows directly. Thus, by Art. 16 we have

$$\frac{d}{dx}(\text{arc cos } u) = \frac{d}{du}(\text{arc cos } u) \cdot \frac{du}{dx},$$

which, by (4), becomes

$$\frac{d}{dx}(\text{arc cos } u) = -\frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if

$$y = \text{arc cos } e^x.$$

SOLUTION. We have

$$y = \text{arc cos } u \quad \text{where} \quad u = e^x.$$

Hence, applying the theorem,

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}e^x}{\sqrt{1 - (e^x)^2}} = -\frac{e^x}{\sqrt{1 - e^{2x}}}.$$

30. The derivative of arc tan u.

Theorem. *The derivative of the function arc tan u, where u is a given function of x, is expressed by the formula*

$$\frac{d}{dx}(\text{arc tan } u) = \frac{\frac{du}{dx}}{1 + u^2}.$$

PROOF. Consider first the function

$$y = \text{arc} \tan x. \tag{1}$$

Then, by Art. 27, we have

$$x = \tan y, \tag{2}$$

and hence, by Art. 21,

$$\frac{dx}{dy} = \sec^2 y.$$

It follows by the lemma of Art. 25 that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}. \tag{3}$$

But from (2) together with formula (3) of page 359 we may write

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

Thus, (3) takes the form

$$\frac{dy}{dx} = \frac{1}{1 + x^2}. \tag{4}$$

The theorem as stated now follows directly. Thus, by Art. 16 we have

$$\frac{d}{dx}(\text{arc} \tan u) = \frac{d}{du}(\text{arc} \tan u) \cdot \frac{du}{dx},$$

which, by (4), becomes

$$\frac{d}{dx} \text{arc} \tan u = \frac{\dfrac{du}{dx}}{1 + u^2}.$$

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if

$$y = \text{arc} \tan \frac{1}{x}.$$

SOLUTION. We have

$$y = \text{arc} \tan u \quad \text{where} \quad u = \frac{1}{x}.$$

Hence, applying the theorem,

$$\frac{dy}{dx} = \frac{\dfrac{d}{dx}\left(\dfrac{1}{x}\right)}{1 + \left(\dfrac{1}{x}\right)^2} = \frac{-\dfrac{1}{x^2}}{1 + \dfrac{1}{x^2}} = -\frac{1}{1 + x^2}.$$

31. The derivative of arc cot u.

Theorem. *The derivative of the function arc cot u, where u is a given function of x, is expressed by the formula*

$$\frac{d}{dx}(\text{arc cot } u) = -\frac{\frac{du}{dx}}{1+u^2}.$$

PROOF. Consider first the function

$$y = \text{arc cot } x. \tag{1}$$

By Art. 27 we then have

$$x = \cot y, \tag{2}$$

and hence, by Art. 22,

$$\frac{dx}{dy} = -\csc^2 y.$$

It follows by the lemma of Art. 25 that

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y}. \tag{3}$$

But from (2) together with formula (4) of page 359, we may write

$$\csc^2 y = 1 + \cot^2 y = 1 + x^2.$$

Thus, (3) takes the form

$$\frac{dy}{dx} = -\frac{1}{1+x^2}. \tag{4}$$

The theorem as stated now follows directly. Thus, by Art. 16 we have

$$\frac{d}{dx}(\text{arc cot } u) = \frac{d}{du}(\text{arc cot } u) \cdot \frac{du}{dx},$$

which, by (4), gives the formula in question.

The proof is thus complete.

Example. Find $\frac{dy}{dx}$ if $y = \text{arc cot}\,(3x^2 - 1)$.

SOLUTION. We have

$$y = \text{arc cot } u \quad \text{where} \quad u = 3x^2 - 1.$$

Hence, applying the theorem,

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(3x^2-1)}{1+(3x^2-1)^2} = -\frac{6x}{9x^4 - 6x^2 + 2}.$$

32. The derivative of arc sec u and of arc csc u.

Theorem. *The derivatives of the functions arc sec u and arc csc u, where u is a given function of x, are expressed respectively by the formulas*

$$\frac{d}{dx}(\text{arc sec } u) = \frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}, \quad \frac{d}{dx}(\text{arc csc } u) = -\frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}.$$

PROOF. Consider first the function

$$y = \text{arc sec } x. \tag{1}$$

By definition we then have

$$x = \sec y, \tag{2}$$

and hence, by Art. 23,

$$\frac{dx}{dy} = \sec y \tan y.$$

It follows by the lemma of Art. 25 that

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}. \tag{3}$$

But from (2) together with formula (3) of page 359, we may write

$$\sec y \tan y = \sec y\sqrt{\sec^2 y - 1} = x\sqrt{x^2 - 1},$$

where the positive sign is used with the radical because, with y restricted to lie either between $-\pi$ and $-\frac{\pi}{2}$ or between 0 and $\frac{\pi}{2}$ (see Fig. 17) the value of $\tan y$ is necessarily positive. Thus, (3) takes the form

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}. \tag{4}$$

The first part of the theorem as stated now follows directly by use of the theorem in Art. 16. Thus, we have

$$\frac{d}{dx}(\text{arc sec } u) = \frac{d}{du}(\text{arc sec } u) \cdot \frac{du}{dx},$$

which, by (4), gives the first of the two formulas.

The proof of the first part of the theorem is thus complete.

The proof of the second part of the theorem, which makes use of Fig. 18 but otherwise is similar to the proof just given for the first part, is omitted.

33. The derivative of arc vers u.

Theorem. *The derivative of the function arc vers u, where u is a given function of x, is expressed by the formula*

$$\frac{d}{dx}\text{arc vers } u = \frac{\frac{du}{dx}}{\sqrt{2u - u^2}}.$$

PROOF. Consider first the function

$$y = \text{arc vers } x. \tag{1}$$

By the definition in Art. 27 we then have

$$x = \text{vers } y = 1 - \cos y, \tag{2}$$

and hence, by the theorems of Arts. 13, 19,

$$\frac{dx}{dy} = \sin y.$$

It follows by the lemma of Art. 25 that

$$\frac{dy}{dx} = \frac{1}{\sin y}. \tag{3}$$

But by (2) we have

$$\sin y = +\sqrt{1 - \cos^2 y} = +\sqrt{1 - (1 - x)^2} = +\sqrt{2x - x^2},$$

where the positive sign is used with the radical because, with y restricted to lie between 0 and π (see Fig. 19), the value of $\sin y$ is necessarily positive. Thus, (3) takes the form

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

The theorem as stated now follows directly, as in former cases, by use of the general theorem of Art. 16.

EXERCISES

Differentiate each of the following functions:

1. $y = \text{arc sin } x^2$. *Ans.* $y' = \frac{2x}{\sqrt{1 - x^4}}$.

2. $y = \text{arc sin } \sqrt{1 - x}$. *Ans.* $y' = \frac{-1}{2\sqrt{x(1 - x)}}$.

3. $y = \text{arc cos } (x^2 - 1)$. *Ans.* $y' = \frac{-2}{\sqrt{2 - x^2}}$.

4. $y = \text{arc cos}\,\dfrac{1}{1-x}$. *Ans.* $y' = \dfrac{-1}{(1-x)\sqrt{x^2-2x}}$.

5. $y = \text{arc tan}\,(1+2x)$. *Ans.* $y' = \dfrac{1}{1+2x+2x^2}$.

6. $y = \text{arc tan}\,\dfrac{x+a}{1-ax}$. *Ans.* $y' = \dfrac{1}{1+x^2}$.

7. $y = \text{arc cot}\,(3x^2-1)$. *Ans.* $y' = -\dfrac{6x}{2-6x^2+9x^4}$.

8. $y = \text{arc cot}\,\dfrac{x+a}{1-ax}$. *Ans.* $y' = -\dfrac{1}{1+x^2}$.

9. $y = \text{arc sec}\sqrt{x+1}$. *Ans.* $y' = \dfrac{1}{2\sqrt{x}(x+1)}$.

10. $y = \text{arc sec}\,\sqrt{x^2+1}$. *Ans.* $y' = \dfrac{1}{x^2+1}$.

11. $y = \text{arc csc}\,\dfrac{1}{2x^2-1}$. *Ans.* $y' = \dfrac{2}{\sqrt{1-x^2}}$.

12. $y = \text{arc csc}\,\dfrac{1}{\sqrt{2x^2-1}}$. *Ans.* $y' = \dfrac{2x}{\sqrt{2(3x^2-1-2x^4)}}$.

13. $y = \text{arc vers}\,(1-2x^2)$. *Ans.* $y' = \dfrac{-4x}{\sqrt{1-4x^4}}$.

14. $y = \text{arc vers}\,\dfrac{2x^2}{1+x^2}$. *Ans.* $y' = \dfrac{2}{1+x^2}$.

15. $y = x\,\text{arc sin}\,x$. *Ans.* $y' = \text{arc sin}\,x + \dfrac{x}{\sqrt{1-x^2}}$.

16. $y = e^x\,\text{arc cos}\,(3x-2)$.

Ans. $y' = e^x\left[\text{arc cos}\,(3x-2) - \left(\dfrac{3}{-1+4x-3x^2}\right)^{\frac{1}{2}}\right]$.

17. $f(x) = \sqrt{a^2-x^2} + a\,\text{arc sin}\,\dfrac{x}{a}$. *Ans.* $f'(x) = \left(\dfrac{a-x}{a+x}\right)^{\frac{1}{2}}$

18. $f(\theta) = \text{arc sin}\,\sqrt{\sin\theta}$. *Ans.* $f'(\theta) = \frac{1}{2}\sqrt{1+\csc\theta}$.

19. $y = x\sqrt{a^2-x^2} + a^2\,\text{arc sin}\,\dfrac{x}{a}$. *Ans.* $f'(x) = 2\sqrt{a^2-x^2}$.

20. $y = \text{arc cos}\,(\log x)$. *Ans.* $f'(x) = \dfrac{-1}{x\sqrt{1-(\log x)^2}}$.

21. $s = \text{arc tan}\,e^\theta$. *Ans.* $s' = \dfrac{e^\theta}{1+e^{2\theta}}$.

22. $s = \text{arc cot}\,e^{\frac{\theta}{2}}$. *Ans.* $s' = \dfrac{-e^{\frac{\theta}{2}}}{2(1+e^\theta)}$.

23. $y = (\text{arc sin}\,x)^2$. *Ans.* $y' = \dfrac{2\,\text{arc sin}\,x}{\sqrt{1-x^2}}$.

24. $y = (\text{arc}\tan x)^m.$ *Ans.* $y' = \dfrac{m(\text{arc}\tan x)^{m-1}}{1+x^2}.$

25. $y = e^{\text{arc}\sin x}.$ *Ans.* $y' = \dfrac{e^{\text{arc}\sin x}}{\sqrt{1-x^2}}.$

26. $u = \text{arc}\tan\left(\dfrac{e^v - e^{-v}}{2}\right).$ *Ans.* $= \dfrac{2}{e^v + e^{-v}}.$

27. $y = \text{arc}\cot\dfrac{a}{x} + \log\sqrt{\dfrac{x-a}{x+a}}.$ *Ans.* $y' = \dfrac{2\,ax^2}{x^4 - a^4}.$

28. $y = x^{\text{arc}\sin x}.$ *Ans.* $y' = x^{\text{arc}\sin x}\left(\dfrac{\text{arc}\sin x}{x} + \dfrac{\log x}{\sqrt{1-x^2}}\right)$

29. Prove that the curve $y = \text{arc}\sin x$ (see Fig. 13) intersects the X-axis at an angle of 45°.

30. Prove that the principal branch of the curve $y = \text{arc}\cos x$ (see Fig. 14) meets the Y-axis at an angle of 135°.

31. Prove that the slope of the principal branch of the curve $y = \text{arc}\tan x$ (see Fig. 15) at any given point (x_1, y_1) upon it is the same as at the point $(-x_1, -y_1)$ situated symmetrically opposite with respect to the origin. State and prove the corresponding facts in the case of the principal branch of the curve $y = \text{arc}\cot x$ (see Fig. 16).

32. Prove that as x increases from -1 to 1 the rate of increase of arc sin x as x passes through any given value is precisely equal to the rate of decrease of arc cos x at the same value of x. State and prove the corresponding fact as regards the rates of change of arc tan x and arc cot x as x increases from $-\infty$ to $+\infty$.

33. Find the derivative of each of the following functions:

(a) $y = \text{arc}\sin\sqrt{1-x^2}$;

(b) $y = \text{arc}\cos\dfrac{1}{1-x^2}$;

(c) $y = \text{arc}\sin\dfrac{x-a}{x+a}$;

(d) $y = \text{arc}\sin\sqrt{\dfrac{x-a}{x+a}}$;

(e) $y = \text{arc}\tan\sqrt{x^2+a^2}$;

(f) $y = \text{arc}\cos\sqrt{x^2+a^2}$;

(g) $y = \text{arc}\tan\dfrac{x}{x+1}$;

(h) $y = \text{arc}\cot\dfrac{x}{x+1}$;

(i) $y = \text{arc}\sec\dfrac{1}{\sqrt{1-x^2}}$;

(j) $y = \text{arc}\csc\dfrac{x}{\sqrt{x^2-1}}$;

(k) $y = \dfrac{\text{arc}\sin x^2}{x}$;

(l) $y = \dfrac{(\text{arc}\sin x)^2}{x}$;

(m) $y = \log \text{arc}\sin x$;

(n) $y = \sin(\text{arc}\sin x)$;

(o) $y = \tan(\text{arc}\cot x)$;

(p) $y = \cot(\text{arc}\tan x)$.

34. A point moves to the right along the curve $y = \text{arc}\tan x$ (Fig. 15) at the uniform speed of 3 inches per second. Find, correct to two decimal places, the x- and the y-components of its velocity at the point where $x = 1$.

34. The differentiation of implicit functions. The special types of functions whose derivatives we have thus far considered all belong to the general class known as **explicit** functions; that is, the dependent variable, y, is connected with the independent variable, x, through an equation which is explicitly solved for y, as in $y = \sqrt{x^2 - 1}$ or in $y = \log x$. It frequently happens, however, that y is connected with x through an equation of a different character, as in $x^2y^3 + 2xy - 1 = 0$ or in $y = \sin (xy)$. In such cases y is said to be an **implicit** function of x. The following examples illustrate the common method of determining the derivative $\frac{dy}{dx}$ in such cases:

Example 1. Find $\frac{dy}{dx}$ if

$$x^2y^3 + 2xy - 1 = 0. \tag{1}$$

SOLUTION. We have

$$\frac{d}{dx}(x^2y^3 + 2xy - 1) = 0, \qquad \text{Th., Art. 13.}$$

$$\frac{d}{dx}(x^2y^3) + \frac{d}{dx}(2xy) - \frac{d}{dx}(1) = 0, \qquad \text{Th. I., Art. 15.}$$

$$x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2) + 2x\frac{d}{dx}(y) + y\frac{d}{dx}(2x) - 0 = 0, \qquad \text{Th. II., Art. 15.}$$

$$x^2 \cdot 3y^2\frac{dy}{dx} + y^3 \cdot 2x + 2x\frac{dy}{dx} + y \ 2 = 0, \qquad \text{Th., Art. 17.}$$

or, collecting terms and transposing,

$$(3x^2y^2 + 2x)\frac{dy}{dx} = -(2xy^3 + 2y).$$

Hence,

$$\frac{dy}{dx} = -\frac{2xy^3 + 2y}{3x^2y^2 + 2x}. \tag{2}$$

Example 2. Find $\frac{dy}{dx}$ if

$$y = \sin (xy). \tag{3}$$

SOLUTION. We have, differentiating the equation member by member,

$$\frac{dy}{dx} = \cos (xy)\frac{d}{dx}(xy) \qquad \text{Th., Art. 19.}$$

$$= \cos (xy)\left[x\frac{dy}{dx} + y\right]$$

$$= x\cos (xy)\frac{dy}{dx} + y \cos (xy).$$

Hence, transposing and collecting terms,

$$\frac{dy}{dx}\cdot [1 - x \cos (xy)] = y \cos (xy).$$

Therefore,

$$\frac{dy}{dx} = \frac{y \cos (xy)}{1 - x \cos (xy)}. \tag{4}$$

Note 1. It is to be observed that the derivative of an implicit function as obtained by the method illustrated in the foregoing examples usually contains the dependent variable y as well as the independent variable x. It is, in fact, difficult or even impossible in general to express the derivative of an implicit function in terms of the independent variable x alone. Theoretically, however, the derivative may be thought of in all cases as a function of x alone, for wherever y occurs in it it is replaceable by the function of x which y represents in the original equation.

Thus, the y in the result (2) may be thought of as replaceable by the expression in x which would result from solving the original relation (1) for y, though actually to effect such a solution would be difficult inasmuch as (1) is a cubic equation in y. Similarly, the y in (4) may be thought of as replaceable by the expression in x which would result by solving (3) for y if it were possible to do so.

Note 2. Not infrequently an implicit function may be converted without difficulty into its corresponding explicit form. Whichever form is used, the derivative is of course the same, or can be identified as the same after suitable reductions.

Thus, the implicit function y defined by the equation

$$x^2 - y^2 = a^2 \tag{5}$$

is equivalent to the explicit function

$$y = \pm \sqrt{x^2 - a^2} \tag{6}$$

since this equation results by solving (5) for y. Incidentally, it appears that in this case y is a double-valued function of x; that is, two values of y correspond to every value of x. If (5) be used to find the derivative, we proceed as in Exs. 1 and 2, thus writing

$$2\,x - 2\,y\frac{dy}{dx} = 0$$

from which we obtain

$$\frac{dy}{dx} = \frac{x}{y}. \tag{7}$$

If, on the other hand, (6) be used, we have by Art. 17

$$\frac{dy}{dx} = \frac{1}{2}\,\frac{\frac{d}{dx}(x^2 - a^2)}{\pm\sqrt{x^2 - a^2}} = \frac{1}{2}\,\frac{2\,x}{\pm\sqrt{x^2 - a^2}} = \frac{x}{\pm\sqrt{x^2 - a^2}}. \tag{8}$$

The fact that the two forms (7), (8) are equivalent appears by replacing the y in (7) by its value as given in (6).

35. Derivatives of higher orders. The derivative of y with respect to x is itself a function of x and is termed the **derivative of the first order,** or simply the **first derivative.** The derivative of the first derivative is called the derivative of the second order, or simply the **second derivative,** and is represented by the symbol $\frac{d^2y}{dx^2}$ (read d second y with respect to x). The derivative of the second derivative is called the derivative of the third order, or simply the **third derivative,** and is represented by the symbol $\frac{d^3y}{dx^3}$; etc. Other symbols for the derivatives of the second, third, orders are y'', y''', $D_x^2 y$, $D_x^3 y$, ; $f''(x), f'''(x)$,

The examples which follow illustrate the usual manner of determining derivatives of higher orders.

Example 1. Find $\frac{d^2y}{dx^2}$ if $y = x^2 + \log x$.

SOLUTION. We have $$\frac{dy}{dx} = 2x + \frac{1}{x}.$$

Hence, differentiating again, $$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}.$$

Example 2. Find $\frac{d^2y}{dx^2}$ if $x^2 - y^2 = a^2$. (1)

SOLUTION. Differentiating both members of the given equation, we obtain

$$2x - 2y\frac{dy}{dx} = 0.$$

Therefore

$$\frac{dy}{dx} = \frac{x}{y}. \tag{2}$$

Differentiating again, remembering that y is a function of x, we obtain

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}. \qquad \text{Th. III, Art. 15.}$$

Substituting for $\frac{dy}{dx}$ its value from (2),

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{x}{y}}{y^2} = \frac{y^2 - x^2}{y^3}.$$

This result may be simplified by substituting for $y^2 - x^2$ the value $-a^2$, as obtained from (1), thus giving as a final answer

$$\frac{d^2y}{dx^2} = -\frac{a^2}{y^3}.$$

EXERCISES

Find the derivative, $\frac{dy}{dx}$, of the implicit function y defined by each of the following equations:

1. $y^2 = 4x$. *Ans.* $y' = \frac{2}{y}$.

2. $x^2 + y^2 = a^2$. *Ans.* $y' = -\frac{x}{y}$.

3. $y^3 - 3y + 2ax = 0$. *Ans.* $y' = \frac{2a}{3(1 - y^2)}$.

4. $x^2y^2 - 3y = 5$. *Ans.* $y' = \frac{2xy^2}{3 - 2x^2y}$.

5. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $y' = -\sqrt{\frac{y}{x}}$.

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $y' = -\sqrt[3]{\frac{y}{x}}$.

7. $x \sin y + y \sin x = 1$. *Ans.* $y' = -\frac{\sin y + y \cos x}{x \cos y + \sin x}$.

8. $y^2 = \cos 2x$. *Ans.* $y' = -\frac{\sin 2x}{y}$.

9. $y^2 \cos x = \sin 3x$. *Ans.* $y' = \frac{3 \cos 3x + y^2 \sin x}{2y \cos x}$.

10. $y^2 = \cos(x + y)$. *Ans.* $y' = -\frac{\sin(x + y)}{2y + \sin(x + y)}$.

11. If $y = x^4 - 3x^3 + x^2 + 2x - 1$, show that $y'' = 12x^2 - 18x + 2$ and $y''' = 24x - 18$.

12. If $f(x) = ax^2 + bx + c$, show that $f'''(x) = 0$.

13. If $y = \log(x + 1)$, show that $\frac{d^4y}{dx^4} = -\frac{6}{(x + 1)^4}$.

14. If $y = \sin ax$, show that $\frac{d^4y}{dx^4} = a^4y$.

15. If $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, show that $y'' = \frac{y}{a^2}$.

16. If $y = (1 + x)(\sin x + \cos x)$, show that

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$$

17. Find y'' if $y^2 + 2y = x^2$. *Ans.* $y'' = \frac{1}{(1 + y)^3}$.
HINT. See Example 2, Art. 35.

18. Find y'' if $y^2 = 4ax$. *Ans.* $y'' = -\frac{4a^2}{y^3}$.

19. Find y'' if $ax^2 + 2bxy + cy^2 = 1$. *Ans.* $y'' = \frac{b^2 - ac}{(bx + cy)^3}$.

20. Find y''' if $y^2 + 2y = x^2$. *Ans.* $y''' = -\dfrac{3x}{(1+y)^5}$.
HINT. See Ex. 17.

21. If $y = \tan(x+y)$, show that

$$y''' = -\frac{2(3y^4 + 8y^2 + 5)}{y^8}.$$

22. If $x^3 + y^3 = 3axy$, show that $y'' = -\dfrac{2a^3xy}{(y^2 - ax)^3}$.

23. By use of the lemma in Art. 25, establish the following general formula:

$$\frac{d^2x}{dy^2} = -\frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3}.$$

24. By use of the theorems in Art. 15, establish the following formula, known as the theorem of Leibnitz, concerning the nth derivative of the product of any two functions u, v:

$$\frac{d^n}{dx^n}(uv) = \frac{d^nu}{dx^n}\cdot v + n\frac{d^{n-1}u}{dx^{n-1}}\cdot\frac{dv}{dx} + \frac{n(n-1)}{\underline{|2}}\frac{d^{n-2}u}{dx^{n-2}}\cdot\frac{d^2v}{dx^2}$$
$$+ \frac{n(n-1)(n-2)}{\underline{|3}}\frac{d^{n-3}u}{dx^{n-3}}\cdot\frac{d^3v}{dx^3} + \cdots + n\frac{du}{dx}\cdot\frac{d^{n-1}v}{dx^{n-1}} + u\frac{d^nv}{dx^n}.$$

25. Use the theorem of Leibnitz to show that the nth derivative of xe^x is equal to $(x+n)e^x$.

26. Find by the methods of Arts. 34 and 35 both y' and y'' for each of the following functions y:

(*a*) $y^3 = x^2$;
(*b*) $y^3 + 2y^2 = x^2$;
(*c*) $ay^2 + by = cx^3$;
(*d*) $b^2x^2 + a^2y^2 = a^2b^2$;
(*e*) $y^2(a - x) = x^2(a + x)$;
(*f*) $x^2y^3 + 3xy = 1$;
(*g*) $x^2y^3 + x^3y^2 = a^3$;
(*h*) $x \sin y = 1$;
(*i*) $x \sin y^2 = 1$;
(*j*) $y \tan y = x$;
(*k*) $y^2 \log(xy) = a$;
(*l*) $y^2 \log(x + y) = a$;
(*m*) $e^{x+y} = xy$;
(*n*) $y = 1 + xe^y$;
(*o*) $e^y + y = e^x + x$;
(*p*) $\tan(xy) = a$.

27. Find y''' for each of the following functions y:

(*a*) $y^3 = x^2$;
(*b*) $y^2 + 2y^3 = x^2$;
(*c*) $b^2x^2 - a^2y^2 = a^2b^2$;
(*d*) $y = \sin(xy)$;
(*e*) $y = \log(x + y)$;
(*f*) $y \sin x = x \sin y$;
(*g*) $x = e^{x+y}$;
(*h*) $x + y = \log y$;
(*i*) $xy = \log y$.

28. Use the theorem of Leibnitz (see Ex. 24) to find

(*a*) the second derivative of $x^2 \sin x$;
(*b*) the third derivative of $e^x \log x$;
(*c*) the fourth derivative of x^2a^x.

CHAPTER IV

MAXIMA AND MINIMA. POINTS OF INFLECTION

36. Definitions. Given any curve which, besides being continuous, presents one or more distinct waves as represented in Fig. 20. The points P, Q, R, S, etc. marking the crests of the waves are known as **maximum points** of the curve, while the points L, M, N, etc. marking the points of greatest depression are called **minimum points.** In case the curve is defined by means of a given equation, as $y = f(x)$, the calculus affords a simple method, as we shall now show, for determining the exact position of these maximum and minimum points. The values of the ordinates AP, CQ, etc., of the maximum points are called the **maximum values** of the function $f(x)$, while the values of the ordinates BL, DM, etc., of the minimum points are called the **minimum values** of $f(x)$.

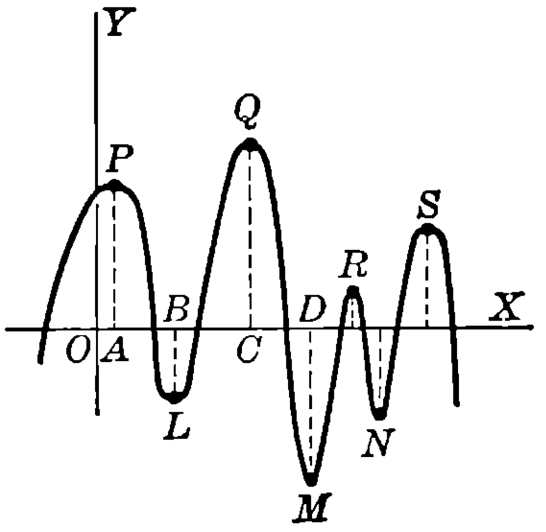


FIG. 20

37. Location of Maximum and Minimum Points. We proceed to establish the following important theorem:

Theorem. *Given a function $f(x)$ whose first and second derivatives are represented respectively by $f'(x)$ and $f''(x)$. If x_1 is a special value of x such that $f'(x_1) = 0$ and $f''(x_1)$ is* ***negative*** *in sign, then x_1 is the abscissa of a* ***maximum*** *point of the curve $y = f(x)$. If, on the other hand, x_1 is such that $f'(x_1) = 0$ and $f''(x_1)$ is* ***positive*** *in sign, then x_1 is the abscissa of a* ***minimum*** *point of the curve $y = f(x)$.*

PROOF. From the fact that $f''(x)$ is the derivative of $f'(x)$ and the fact that, by hypothesis, $f''(x_1)$ is negative in sign, it follows from the theorem of Art. 10 that as x passes by continually increasing values through the fixed value x_1 the corresponding values of $f'(x)$ are continually decreasing. But this in turn implies, by Art. 9, that in the neighborhood of the point P whose abscissa is x_1 the tangent lines AB, CD, EF, GH, IJ, etc., that can be drawn to the curve $y = f(x)$ have continually decreasing slopes. Moreover, the further assumption of the

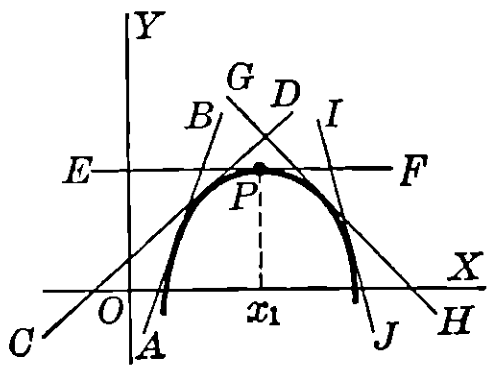


FIG. 21

theorem that $f'(x_1) = 0$ implies that at the point P in question the slope of the curve is equal to zero; that is, the tangent EF at this point is parallel to the X-axis. Therefore, the general shape of the curve in the neighborhood of P must be as indicated in Fig. 21, the point P itself being necessarily a maximum point.

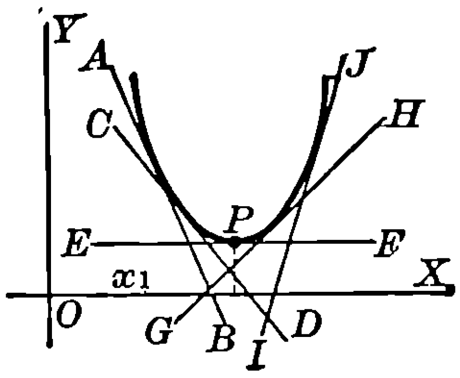


FIG. 22

If, on the other hand, we have $f'(x_1) = 0$ but $f''(x_1)$ positive in sign, similar reasoning based upon Arts. 9, 10 shows that as x increases through the fixed value x_1 the tangent lines at the corresponding points upon the curve have continually increasing slopes, while the value of the slope at the special point P whose abscissa is x_1 is, as before, equal to zero. Hence the general shape of the curve in the neighborhood of P must be as indicated in Fig. 22, the point P itself being necessarily a minimum point.

The proof is thus complete.

Note 1. If a curve $y = f(x)$ has a maximum or a minimum point where $x = x_1$, then necessarily we have $f'(x_1) = 0$, for it is only then that the tangent line at the point in question can be parallel to the X-axis. Hence, the single equation $f'(x_1) = 0$ may be described as a *necessary* condition for either a maximum or a minimum. This single condition, however, is not *sufficient*, for it is possible to have $f'(x_1) = 0$ and yet not have either a maximum or a minimum point present, as will appear in Chap. VIII. To be assured of the existence of such a point one must have, besides the condition $f'(x_1) = 0$, the further condition mentioned in the theorem relative to the sign of $f''(x_1)$, or some other supplementary condition.

Note 2. If, in the language of the theorem, x_1 is such that $f''(x_1) = 0$, the theorem cannot be applied, for $f''(x_1)$ cannot then be regarded as having any definite sign. Such exceptional cases will be considered subsequently. See Art. 41.

Example 1. Determine the maximum and the minimum points of the curve whose equation is

$$y = x^3 - 6x^2 + 9x + 1. \tag{1}$$

SOLUTION. In the language of the theorem we have

$$f(x) = x^3 - 6x^2 + 9x + 1$$

and therefore

$$f'(x) = 3x^2 - 12x + 9,$$
$$f''(x) = 6x - 12. \tag{2}$$

The values of x for which $f'(x)$ becomes equal to zero are therefore the roots of the quadratic equation

$$3x^2 - 12x + 9 = 0, \quad \text{or} \quad x^2 - 4x + 3 = 0. \tag{3}$$

Solving (3) by any one of the usual methods, its two roots are found to be $x = 1$ and $x = 3$.

When $x = 1$, we have from (2)

$$f''(1) = 6 \;\; 1 - 12 = -6,$$

and when $x = 3$, we have

$$f''(3) = 6 \cdot 3 - 12 = 18 - 12 = +6.$$

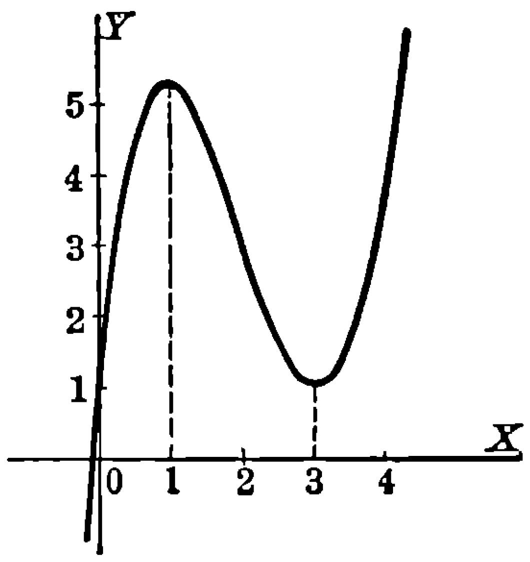


FIG. 23

Applying the theorem, it thus appears that the curve (1) has a maximum point whose abscissa is $x = 1$ and a minimum point whose abscissa is $x = 3$. To find the ordinates of these points, we need only substitute these values of x in (1) and note the resulting values of y. Thus, when $x = 1$, we find $y = 5$, and when $x = 3$, we find $y = 1$.

In summary, the curve (1) has a maximum point at (1, 5) and a minimum point at (3, 1), as shown in Fig. 23. Moreover, by Note 1 there can be no other such points upon the curve, since it is only at these that $f'(x) = 0$.

Example 2. Find whether the function

$$y = \frac{\log x}{x} \tag{4}$$

has any maximum or minimum values and, if so, determine them.

SOLUTION. Proceeding as in Example 1, we have

$$f(x) = \frac{\log x}{x}$$

and therefore

$$f'(x) = \frac{1 - \log x}{x^2} \quad \text{and} \quad f''(x) = \frac{-3 + 2\log x}{x^3}. \tag{5}$$

The equation $f'(x) = 0$ thus becomes

$$1 - \log x = 0 \qquad \text{or} \qquad \log x = 1,$$

and this equation has the single root $x = e$. When x has this value, the second derivative, as obtained in (5), is seen to have the negative value $-\frac{1}{e^3}$. Hence, applying the theorem, the function (4) has a maximum value when

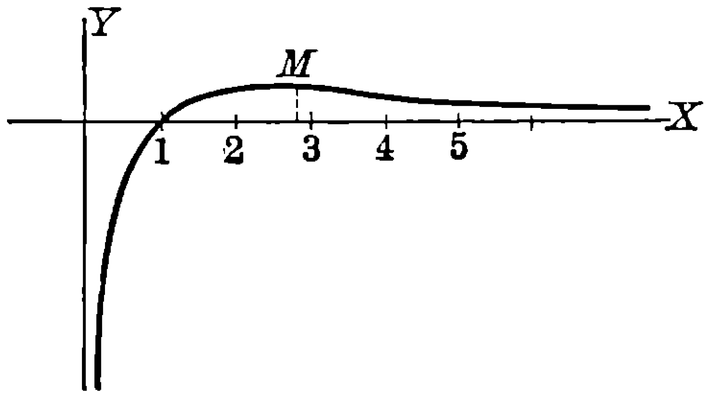


FIG. 24

$x = e$ and it has no other maximum or minimum values whatever. For this value of x the value of y as computed from (4) is $\frac{1}{e}\log e = \frac{1}{e}$. This is, therefore, a maximum value of y, and y takes on no other maximum or minimum values. This result is illustrated in Fig. 24, where the graph of the function (4) is shown, the point M being the one whose abscissa is $x = e$. Note that the curve possesses no other point at which y takes on either a maximum or a minimum value.

EXERCISES

By use of the theorem of Art. 37 determine the maximum and the minimum points upon each of the following curves:

1. $y = x^3 - 3x^2 - 9x$. *Ans.* Max. at $(-1, 5)$; min. at $(3, -27)$.

2. $y = 8x^3 + 9x^2 - 6x + 2$. *Ans.* Max. at $(-1, 9)$; min. at $(\frac{1}{4}, \frac{19}{16})$.

3. $y = x(a + x)^2(a - x)^2$. *Ans.* Max. at $(-a, 0)$ and $\left(\frac{\sqrt{5}}{5}a, \frac{16\sqrt{5}}{125}a^5\right)$.

Min. at $(a, 0)$ and $\left(-\frac{\sqrt{5}}{5}a, -\frac{16\sqrt{5}}{125}a^5\right)$.

4. $y = (x + 1)^{\frac{2}{3}}(x - 5)^2$. *Ans.* Max. at $(\frac{1}{2}, \frac{81}{8}\sqrt[3]{18})$; min. at $(5, 0)$.

5. $y = \frac{2x}{1 + x^2}$. *Ans.* Max. at $(1, 1)$; min. at $(-1, -1)$.

6. $y = \frac{a^2}{x} + \frac{b^2}{a - x}$, where a is greater than b and both are positive.

Ans. Min. at $\left(\frac{a^2}{a + b}, \frac{(a + b)^2}{a}\right)$; max. at $\left(\frac{a^2}{a - b}, \frac{(a - b)^2}{a}\right)$.

7. $y = \frac{x}{(1 + x^2)^{\frac{5}{2}}}$. *Ans.* Max. at $(\frac{1}{2}, \frac{16}{125}\sqrt{5})$; min. at $(-\frac{1}{2}, -\frac{16}{125}\sqrt{5})$

8. $y = \frac{(x - a)(b - x)}{x^2}$. *Ans.* Max. at $\left(\frac{2ab}{a + b}, \frac{(a - b)^2}{4ab}\right)$.

9. $y = \sin 2x - x$. *Ans.* Max. at $\left(\frac{\pi}{6}, \frac{3\sqrt{3} - \pi}{6}\right)$; min. at $\left(-\frac{\pi}{6}, \frac{\pi - 3\sqrt{3}}{6}\right)$.

10. $y = \frac{\cos x}{1 + \cot x}$. *Ans.* Max. at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{4}\right)$; min. at $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{4}\right)$.

11. $y = \frac{\sin x}{1 + \tan x}$. *Ans.* Max. at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{4}\right)$; min. at $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{4}\right)$.

12. $y = \sin x + \cos 2x$. *Ans.* Max. at $\left(\arcsin\frac{1}{4}, \frac{9}{8}\right)$; min. at $\left(\frac{\pi}{2}, 0\right)$.

13. $y = x^x$. *Ans.* Min. at $\left(\frac{1}{e}, e^{-\frac{1}{e}}\right)$.

14. $y = \dfrac{\log \log x}{\log x}$. *Ans.* Max. at $\left(e^{e}, \dfrac{1}{e}\right)$.

15. $y = ae^{kx} + be^{-kx}$. *Ans.* Min. at $\left(\dfrac{1}{2k} \log \dfrac{b}{a}, 2\sqrt{ab}\right)$.

16. $y = 2e^{-\frac{x}{4}} \sin 2x$.

Find the maximum and the minimum values (if any) of each of the following functions:

17. $x^3 - 9x^2 + 15x - 3$. *Ans.* Max. val. $= 4$; min. val. $= -28$.

18. $3x^5 - 125x^3 + 2160x$. *Ans.* Max. vals. $= -3712$ and 3834;
min. vals. $= -3834$ and 3712.

19. $(1 + x^{\frac{2}{3}})(7 - x)^2$. *Ans.* Max. val. $= 72$; min. vals. $= 0$ and 49.

20. $\dfrac{(a - x)^3}{a - 2x}$. *Ans.* Min. val. $= \dfrac{27a^2}{32}$.

21. $\dfrac{1 - x + x^2}{1 + x - x^2}$. *Ans.* Min. val. $= \dfrac{3}{5}$.

22. $\dfrac{x^2 - 7x + 6}{x - 10}$. *Ans.* Max. val. $= 25$; min. val. $= 1$.

23. $\dfrac{3x^2 - a^2}{(a^2 + x^2)^3}$. *Ans.* Max. val. $= \dfrac{1}{4a^4}$; min. val. $= -\dfrac{1}{a^4}$.

24. $2 \tan x - \tan^2 x$. *Ans.* Max. val. $= 1$.

25. $2 \tan x + \sec^2 x$. *Ans.* Min. val. $= 0$.

26. $\log \cos x$. *Ans.* Max. val. $= 0$.

27. $\sin^3 x \cos x$. *Ans.* Max. val. $= \frac{3}{16}\sqrt{3}$; min. val. $= -\frac{3}{16}\sqrt{3}$.

28. $^{\frac{1}{\ }}$ *Ans.* Max. val. $= e^{\frac{1}{e}}$.

29. $\dfrac{x}{\log x}$. *Ans.* Min. val. $= e$.

30. $x - e^x$. *Ans.* Max. val. $= -1$.

31. *Ans.* Max. val. $= \dfrac{1}{e}$.

32. $e^x \cos x$. *Ans.* Max. val. $= \dfrac{\sqrt{2}}{2} e^{\frac{\pi}{4}}$; min. val. $= -\dfrac{\sqrt{2}}{2} e^{\frac{5\pi}{4}}$.

33. Determine the maximum and the minimum points upon each of the following curves.

(*a*) $y = 2x^3 - 9x^2 + 12x - 3$;

(*b*) $y = x^3 - 3x^2 - 9x + 6$;

(*c*) $y = ax + \dfrac{b}{x}$;

(*d*) $y = \dfrac{x}{1 + x^2}$;

(*e*) $y = \dfrac{x^2}{1 + x^2}$;

(*f*) $y = (1 - x^{\frac{2}{3}})^3$;

(*g*) $y = \dfrac{(x - a)(x - b)}{x^2}$;

(*h*) $y = \sin x(1 + \cos x)$;

(*i*) $y = \tan x + 3 \cot x$;

(*j*) $y = e^x + e^{-x}$.

38. Abridged Method. The values of x that render a given function $f(x)$ a maximum or a minimum may frequently be determined directly from a knowledge of the roots of the equation $f'(x) = 0$. In other words, the labor of finding the second derivative $f''(x)$ and the subsequent consideration of its sign, as called for by the theorem of Art. 37, may frequently be dispensed with. This fact, which is particularly important in connection with applied problems, is illustrated in the following examples:

Example 1. Two sides of a triangle being given, what should be the value of the angle between them in order that the area of the triangle may be a maximum?

SOLUTION. Let the given sides be represented by a and b and let x be the (variable) angle between them. We first observe (see Fig. 25) that as x ranges from very small values up to values near to 180° the area of the corresponding triangle AOP, while always changing continuously, at first increases but eventually decreases, becoming zero at 180° It follows that the area of AOP has one maximum value but no minimum value as x ranges in the manner just indicated. It remains but to determine the exact value of x at which this maximum area occurs.

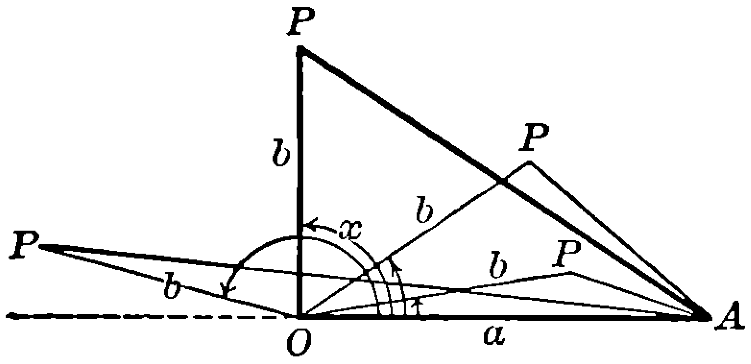


FIG. 25

Representing the area of the triangle AOP by y, we have by trigonometry

$$y = \tfrac{1}{2} ab \sin x. \tag{1}$$

The derivative of this function of x is $\frac{1}{2}ab \cos x$. Hence (see Note 1, Art. 37) the desired value of x is necessarily the root lying between 0° and 180° of the equation

$$\tfrac{1}{2} ab \cos x = 0,$$

or

$$\cos x = 0.$$

But the value of x between 0° and 180° which satisfies this equation is, by trigonometry, $x = 90°$ We therefore conclude that the triangle has its area a maximum when the two given sides include a right angle.

Note. In the foregoing simple example, the answer to which may be surmised beforehand by inspection of Fig. 25, the quantity y which is to be made a maximum is readily expressible in terms of the variable x, as in (1), thus making the problem readily solvable. This should now be contrasted with Example 2 wherein the chief difficulty is merely that of obtaining the expression for V in terms of the altitude x of the cone.

Example 2. Find the altitude of the right cone of minimum volume that can be circumscribed about the sphere whose radius is r.

SOLUTION. Let x represent the (variable) altitude DC of the circumscribed cone. Then x can take any value greater than DE, or $2r$. Moreover, it appears by inspection that as x ranges from values near to $2r$ through larger values the corresponding circumscribed cone ABC at first has a very large volume which continually diminishes for a time but eventually begins to increase and finally increases without limit as x becomes very large. It follows that the volume of the cone ABC has one minimum value but no maximum value as x ranges upward from its smallest value, $2r$. It remains but to determine the exact value of x at which this minimum volume occurs.

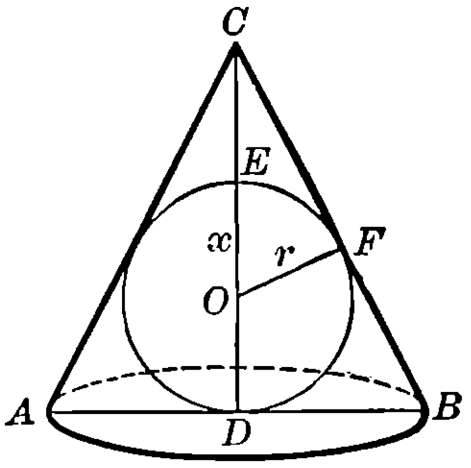


FIG. 26

Representing the volume of the cone ABC by V, we have by solid geometry

$$V = \tfrac{1}{3}\pi\overline{DB}^2 \times DC. \tag{1}$$

Moreover, since the right triangles CDB and CFO are similar (having the angle OCF in common) we may write

$$DB \quad DC = OF : FC.$$

Hence,
$$DB = \frac{DC \times OF}{FC} = \frac{x \cdot r}{FC} = \frac{rx}{FC}. \tag{2}$$

But

$$FC = \sqrt{\overline{OC}^2 - \overline{OF}^2} = \sqrt{(CD - OD)^2 - \overline{OF}^2} = \sqrt{(x - r)^2 - r^2} = \sqrt{x^2 - 2rx}.$$

Hence,
$$DB = \frac{rx}{\sqrt{x^2 - 2rx}}.$$

Substituting the value of DB in (1), recalling also that $DC = x$, we obtain

$$V = \frac{\pi r^2 x^3}{3(x^2 - 2rx)} = \frac{\pi r^2}{3} \quad \frac{x^2}{x - 2r}. \tag{3}$$

Thus V becomes expressed as a function of the single variable x.

Differentiating, we obtain

$$\frac{dV}{dx} = \frac{\pi r^2}{3} \quad \frac{(x - 2r)\,2x - x^2}{(x - 2r)^2} = \frac{\pi r^2}{3} \quad \frac{x^2 - 4rx}{(x - 2r)^2}.$$

The value of x at which V attains its minimum value is therefore necessarily determined (see Note 1, Art. 37) by the equation

$$x^2 - 4rx = 0. \tag{4}$$

The roots of (4) are $x = 0$ and $x = 4r$, and of these only the root $x = 4r$ pertains to the problem since, as stated before, x must be greater than $2r$.

The cone will therefore have a minimum volume when its altitude is equal to four times the radius of the sphere.

39. Special methods. The labor involved in solving problems in maxima and minima by the direct methods of Arts. 37, 38 can frequently be greatly reduced by introducing two (or more) variables into the problem and then using some special ingenious method to arrive at the desired result. While no general rule can be stated in this connection, the following example will suffice to illustrate the use of such methods.

Example. What are the most economical proportions to use in making an ordinary cylindrical pail from sheet metal?

SOLUTION. The question is equivalent to asking what proportions should be used in a right circular cylinder of given volume, V, in order that the sum of the lateral area and the area of the base may be a minimum.

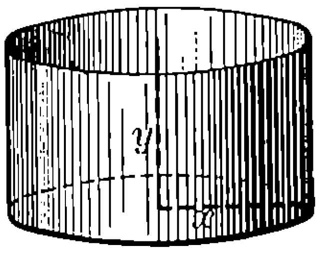


FIG. 27

Let x represent the (variable) radius of the base and let y represent the (variable) height. Then, by solid geometry, we have

$$V = \pi x^2 y. \tag{1}$$

Also, if A represents the sum of the lateral area and the area of the base, we have

$$A = 2\pi xy + \pi x^2. \tag{2}$$

Differentiating equations (1) and (2) member by member with respect to x, recalling that by hypothesis V is a constant and that $\frac{dA}{dx} = 0$ when A is a minimum, we obtain the following two equations which must coexist between x, y, and $\frac{dy}{dx}$ when A is at its minimum value:

$$\begin{cases} 0 = \pi x^2 \dfrac{dy}{dx} + 2\pi xy, & (3) \\ 0 = 2\pi x \dfrac{dy}{dx} + 2\pi y + 2\pi x. & (4) \end{cases}$$

From (3) we obtain

$$\frac{dy}{dx} = -\frac{2y}{x},$$

and substituting this expression for $\frac{dy}{dx}$ in (4), we obtain after simplification and rearrangement of terms

$$y = x.$$

The pail will therefore have the most economical proportions when its height is equal to the radius of its base.

EXERCISES

The following exercises are divided into three groups: I, II, III. For the exercises in Group I the direct method explained in Art. 38 may well be used; for those in Group II it will be found advantageous to use the special method explained in Art. 39; for those in Group III the student should decide for himself which method is best. It is preferable to work several problems from each group rather than confine the attention to any one group.

GROUP I

1. Divide the number 16 into two parts such that their product shall be a maximum. *Ans.* 8, 8.

2. Divide the number a into two parts such that the sum of their squares shall be a minimum. *Ans.* $\frac{a}{2}, \frac{a}{2}$.

3. Divide the number a into two parts such that the sum of the double of the one and the square of the other shall be a minimum. *Ans.* $a - 1$, 1.

4. What number exceeds its square by the greatest amount? *Ans.*

5. It is desired to divide the number ke into a number of equal parts in such a way that the product of all the parts shall be a maximum, it being understood that k is any positive integer and that e is the base of the Naperian system of logarithms. Show that for this purpose each part must be taken equal to e; that is, there must be k parts.

HINT. Let x be the number of parts. Then any one part is equal to $\frac{ke}{x}$ and the product of all the parts is $\left(\frac{ke}{x}\right)^x$

6. A boatman is at A which is 3 miles from the nearest point B on a straight shore LM. He wishes to reach in minimum time a point C situated on shore 6 miles from B. How far from C should he land if he can row at the rate of 4 miles an hour and can walk at the rate of 5 miles an hour? *Ans.* 2 miles.

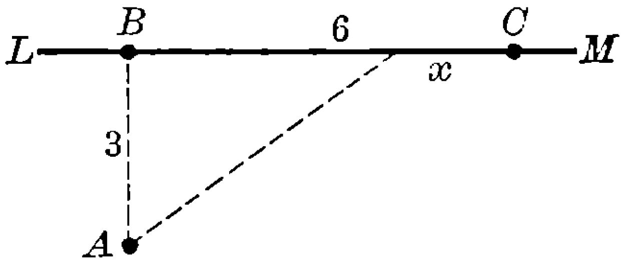


FIG. 28

HINT. In all cases of uniform speed, the distance travelled equals the time occupied multiplied by the rate.

7. A two-branched figure in the form of a letter Y and having a total height of 16 inches and a total width across the top of 12 inches is to be made in such a way that the total length of the figure shall be a minimum. How long should the vertical stem be taken? *Ans.* $16 - 2\sqrt{3}$ inches.

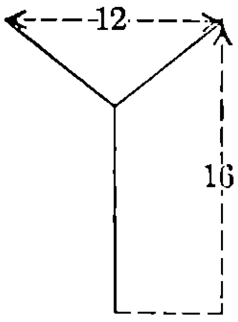


FIG. 29

8. A length of wire is cut into two portions which are bent into the shapes of a circle and a square respectively. Show that if the sum of the areas obtained is the least possible, the side of the square will be equal to the diameter of the circle.

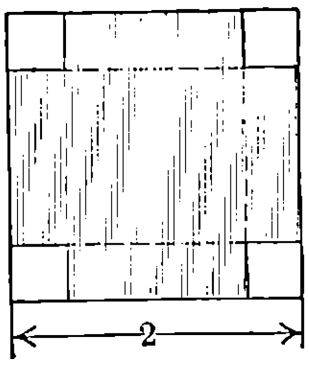


FIG. 30

9. It is desired to make an open box of greatest possible volume from a square piece of cardboard whose sides are each 2 feet long by cutting equal small squares out of the corners and then folding up the remaining piece so as to form the sides of the box. How long should each side of the small squares be made? *Ans.* 4 in.

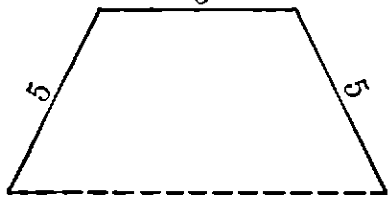


FIG. 31

10. If three sides of a trapezoid are each equal to 5 inches, how long must the remaining side be taken in order that the trapezoid shall have a maximum area? *Ans.* 10 inches.

11. It is desired to cut off the two shaded portions of the triangle ABC in such a manner that the remaining parallelogram $APQR$ shall have the greatest possible area. Show that for this purpose each side of the triangle should be cut at its middle point.

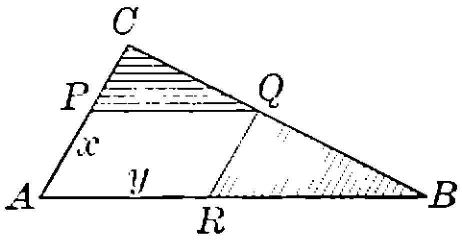


FIG. 32

HINT. Letting $AP = x$ and $AR = y$, we have by trigonometry *area of parallelogram* $APQR = xy \times \sin A$. Now note certain similar triangles, thus obtaining y in terms of x and the lengths of the sides of the triangle which, for convenience, may be represented by a, b, c. Compare Example 2, Art. 38.

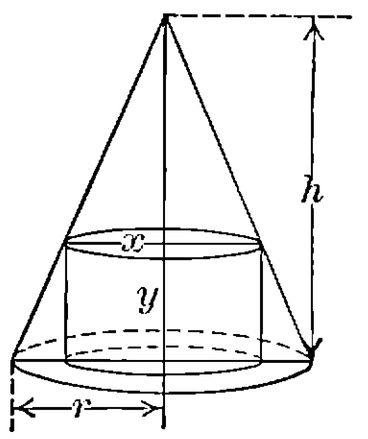


FIG.

12. Show that the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone is equal to one third of the altitude of the cone.

HINT. In the notation of the figure, the volume of the cylinder is $\pi x^2 y$. By the consideration of similar triangles show that this may be written in the form

$$\frac{\pi r^2}{h^2}\, y(h - y)^2.$$

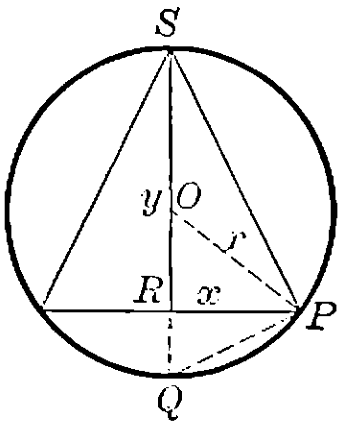


FIG. 34

13. Show that the isosceles triangle of maximum area that can be inscribed in a given circle is equilateral.

HINT. In the figure note that since the angle SPQ is a right angle (being inscribed in a semicircle), we have by geometry $\overline{RP}^2 = RS \times RQ$. For convenience use x to represent RP, y to represent RS, and note that we then have $RQ = 2r - y$. Hence show that the area of the triangle may be expressed in the form $y^{\frac{3}{2}}(2r - y)^{\frac{1}{2}}$.

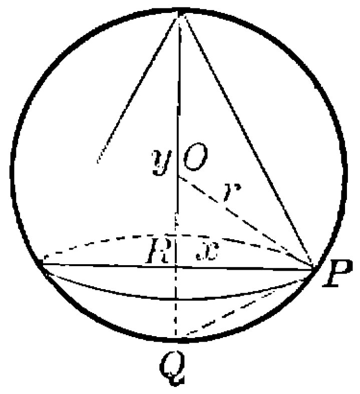


FIG.

14. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a given sphere is equal to four thirds of the radius.

HINT. Make observations similar to those in the hint of Ex. 13, showing that, in the notation of Fig. 35 the volume in question may be expressed in the form

$$\frac{\pi}{3}\, y^2(2r - y), \quad \text{where} \quad y = SR.$$

15. A rectangular flower garden is to contain 20,000 square feet and is to be laid out in such a manner that there shall be a border of shrubs 10 feet wide along each of the longer sides and a similar border 20 feet wide along each of the shorter sides. Determine the dimensions of the smallest rectangle of land necessary to contain both garden and shrubs. *Ans.* 240 feet by 120 feet.

16. A ladder of length d stands against the side of a house. If P is the point on the ground which is distant a from the house, show that the vertical distance PQ from P to the ladder will be greatest when the foot of the ladder is at the distance $\sqrt[3]{ad^2}$ from the house.

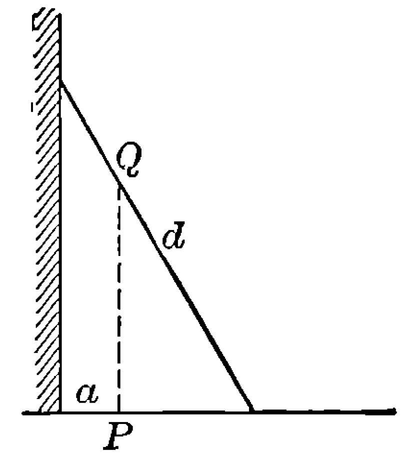


FIG. 36

17. A fence $3\frac{3}{8}$ feet high is 8 feet from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside the fence. Compare Ex. 16. *Ans.* $15\frac{5}{8}$ feet.

18. Prove that arc tan $f(x)$ can have a maximum or a minimum value only at such values of x as render $f(x)$ itself a maximum or a minimum, $f(x)$ being any function possessing a derivative $f'(x)$ for every value of x.

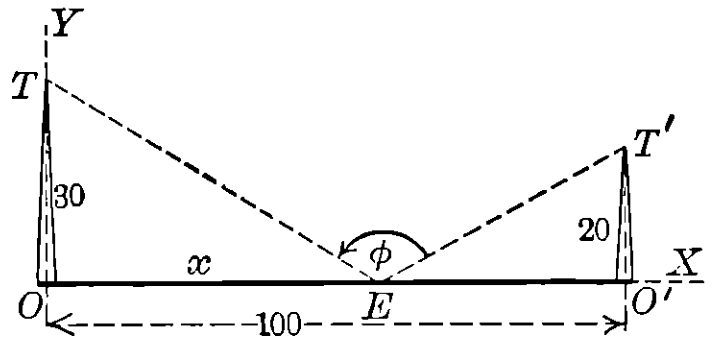


FIG. 37

19. Two vertical poles OT and $O'T'$, whose lengths are respectively 30 feet and 20 feet, stand 100 feet apart upon a horizontal plane. In order that the angle ϕ between the tops T, T' as viewed from various points E in the line OO' may be greatest, show that the distance OE should be equal to the smaller of the two roots of the quadratic equation

$$x^2 - 600\,x + 29400 = 0$$

and hence should be $300 - 10\sqrt{606}$, or 53.83 feet, approximately.

HINT. Take O as origin of a system of coördinates, with OO' as X-axis and OT as Y-axis. Let x be the (variable) distance OE. Then the coördinates of the points T, E, T' are respectively (0, 30); (x, 0); (100, 20). Find the slopes of the lines ET, ET' and then show that

$$\tan\phi = \frac{10\,x - 3000}{x^2 - 100\,x + 600},$$

$$\phi = \text{arc tan}\,\frac{10\,x - 3000}{x^2 - 100\,x + 600}.$$

Now make use of the result in Ex. 18.

20. A tapestry 7 feet in length is hung on a wall so that its lower edge is 9 feet above the observer's eye. At what distance from the wall should he stand in order to obtain the most favorable view? *Ans.* 12 feet.

HINT. The vertical angle subtended by the tapestry at the eye must be at a maximum. Compare Ex. 19.

Group II

21. Show that of all rectangles having a given area the one of least perimeter is a square.

22. Show that of all rectangles having a given perimeter the one of greatest area is a square.

23. A rectangular garden plot is to be inclosed alongside a house, using 32 feet of wire fencing on the three exposed sides. What should be the dimensions used in order that the inclosed area may be as great as possible? *Ans.* 16 ft. by 8 ft.

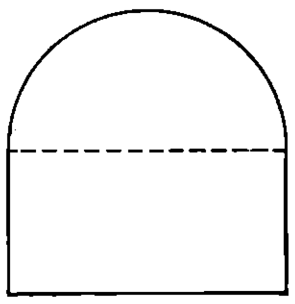


Fig. 38

24. Prove that a window of the shape here shown (Norman window) and having a given perimeter will admit the most light when the height of its rectangular base equals the radius of its semicircular top.

25. A window in the form of a rectangle surmounted by an equilateral triangle is to have a total perimeter of 35 feet. Find the dimensions of the rectangle in order that such a window may admit the most light. *Ans.* 8.2 feet by 5.2 feet.

26. A right circular cylinder of given volume is to be so shaped that its total surface (both bases plus the convex surface) shall be as small as possible. Show that for this purpose the height should be taken equal to twice the radius of the base. Compare Example, Art. 39.

27. Show that the rectangle of maximum area that can be inscribed in a given circle is a square.

28. Show that the dimensions of the rectangle of maximum area that can be inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are in the ratio $a : b$.

29. Show that the area of the smallest ellipse that can be circumscribed about a rectangle is to the area of the rectangle in the ratio $\pi : 2$.

30. Show that for a given amount of convex surface the right circular cone of maximum volume will have an altitude and radius of base in the ratio $\sqrt{2} : 1$.

31. Given the total surface and the altitude of a rectangular parallelopiped. Show that if the volume is to be a maximum, the remaining dimensions must be equal.

32. A water tank is to be constructed having a square base and open top, the sides being vertical. It is to contain 192 cubic yards. If the cost (including labor) of the sides is \$4 per square yard and of the bottom \$3 per square yard, what should be the dimensions in order that the cost may be a minimum, and what will the cost be? *Ans.* Side of base = 8 yards, height = 3 yards. Cost = \$576.

33. A silo is to be made in the form of a cylinder with a floor and a hemispherical roof all of the same material. Determine the most economical proportions.

34. The sum of the surfaces of a sphere and a cube being given, show that the sum of their volumes will be least when the diameter of the sphere is equal to the edge of the cube.

35. If the combined length and girth of a parcel post package is to be 60 inches, find the maximum volume, the package being a box of square cross section.

36. Work Ex. 8 by the method illustrated in Art 39.

GROUP III

37. It is shown in mechanics that the horizontal range OA (measured in feet) which will be attained by a projectile when fired at O is given by the following formula in case the resistance of the air is neglected:

$$OA = \frac{v_1^2 \sin 2\phi}{g},$$

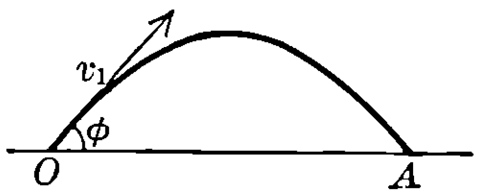


FIG. 39

where v_1 is the initial velocity of firing (measured in feet per second), where ϕ is the angle of projection, and where g is a constant whose value is 32.2, approximately. Show that the greatest possible range is obtained by taking $\phi = 45°$

38. The time (measured in seconds) which it takes a small heavy ball to roll down an inclined plane in case the resistance of the air is neglected is given by the formula

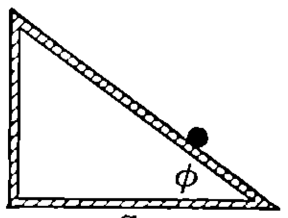


FIG. 40

$$T = 2\sqrt{\frac{a}{g \sin 2\phi}},$$

where ϕ is the angle of inclination of the plane and a is the length of the base (measured in feet) and where g is the constant mentioned in Ex. 37. Hence show that, for a given length of base, the time of descent will be a minimum on a plane of 45° inclination.

39. An arc light L is to be placed directly over the center O of a circular grass-plot a feet in radius. It being given that the intensity of the illumination at any point P in the circle varies directly as the sine of the angle LPO and inversely as the square of the distance LP, find how high the light should be placed above O in order that a narrow path along the circumference of the grass-plot may be most brilliantly lighted. *Ans.* $\frac{a}{\sqrt{2}}$.

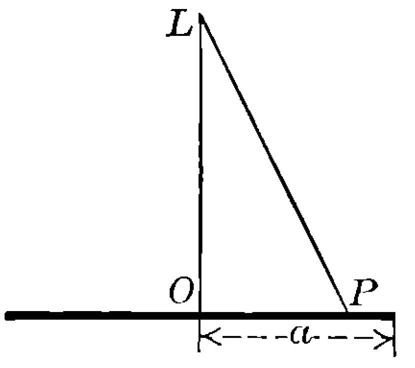


FIG. 41

40. Two vertical poles of heights a and b respectively are standing on a horizontal plane c units apart. Show that the point on the line joining their bases such that the sum of the squares of the distances from this point to the tops of the poles is a minimum is situated halfway between the bases.

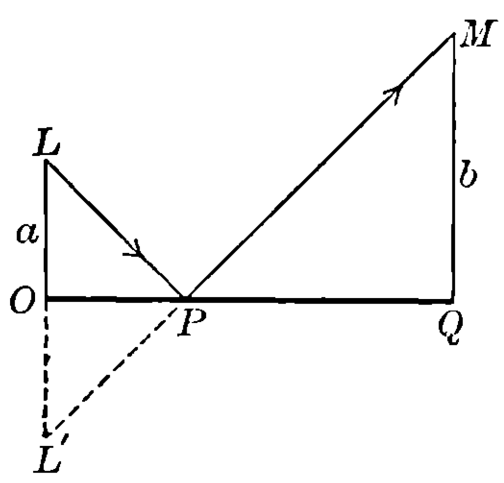


FIG. 42

41. If, in Ex. 40, it is desired to find the point P on the line OQ joining the bases such that the sum of the distances from this point to the tops of the poles is a minimum, show that it is the point P such that $OP : PQ = a : b$. Hence show that the position of P may be constructed geometrically by prolonging LO by its own length to L' and then joining L' to M, the point P where $L'M$ intersects OQ being the point desired.

Note. It may be observed in Fig. 42 that the angles OPL and QPM are equal. Hence the following principle in optics: *Whenever a ray of light is emitted from a source L in such a way as to pass through a given point M after reflection from a plane mirror OQ, it follows the shortest possible path between the two points.*

42. A person L is in a field at a distance a from a fence OQ and he wishes to go to a point M situated at a distance b into the next field and at a distance c to the right of this former position. If he can travel at the rate of v_1 feet per minute in the first field and at the rate v_2 feet per minute in the second field, show that he should cross the fence at the point P such that $\frac{\sin \phi}{\sin \theta} = \frac{v_1}{v_2}$, the angles ϕ and θ being respectively the angles made by LP and MP with the direction perpendicular to OQ.

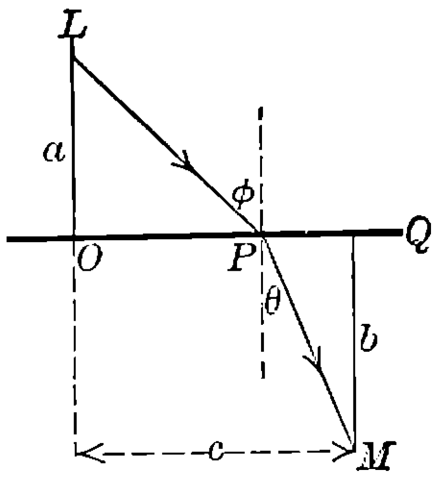


FIG. 43

Note. It is shown experimentally in physics that if a ray of light emanates from a point L and, after traveling through one medium such as air, strikes another transparent medium such as water, the ray is bent at the point P of incidence, and in such a way as to follow the law just described; that is,

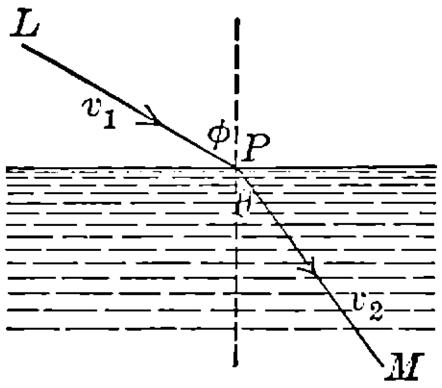


FIG. 44

$$\frac{\sin \phi}{\sin \theta} = \frac{v_1}{v_2},$$

where v_1 and v_2 now represent the velocities of light in the two respective media. The law is known as "*Snell's Law of Refraction.*" It follows from this fact and the foregoing statements that a refracted ray LPM is always bent in such a manner that the light passes from L to M in the least possible time.

43. An open trough having a cross section in the form of an arc of a circle is to be constructed by bending a long rectangular strip of metal of breadth $2\,a$. Show that in order that the trough may have maximum capacity the radius used should be equal to $\frac{2\,a}{\pi}$; that is, the arc will form a semicircle.

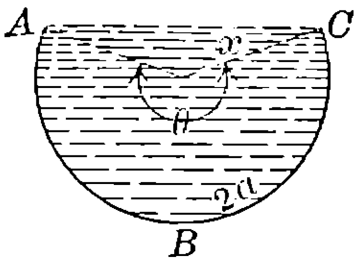


FIG. 45

HINT. Let x be the (variable) radius and θ the (variable) central angle measured in radians. Then, by geometry

$$x\theta = 2\,a, \text{ or } x = \frac{2\,a}{\theta}.$$

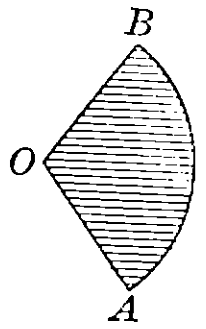


FIG. 46

Now show that the total area of the segment ABC is expressible in the form

$$\frac{2\,a^2}{\theta^2}(\theta - \sin \theta).$$

44. Show that of all circular sectors having the same perimeter the one of greatest area is that whose arc is double its radius.

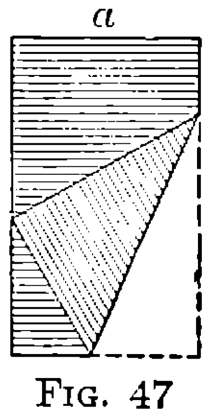


FIG. 47

45. The lower corner of a leaf whose width is a is folded over so as just to reach the inner edge of the page. Find the width of the part folded over (a) when the length of the crease is a minimum; (b) when the area folded over is a minimum.

46. Find the length of the longest rod that can be passed (without tilting to one side) up a chimney where it enters an open fire-place, the width of the chimney being a and the height of the opening above the floor being h. Compare Ex. 16.

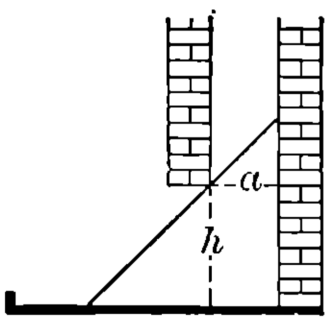


FIG. 48

47. It is shown in solid geometry that the area of the zone of a sphere of radius R that will be visible at the distance x from the center is equal to $\frac{2\pi R^2(x-R)}{x}$. Hence, show that on the line OO' joining the centers of two spheres whose radii are respectively R and r the point at which the most spherical surface is visible (counting both spheres) is at the distance $\frac{aR^{\frac{3}{2}}}{R^{\frac{3}{2}}+r^{\frac{3}{2}}}$ from O, a being the distance OO' between centers.

48. Show that the sector that should be removed from a given circle in order that the remaining area may form the curved surface of a cone of maximum volume has as its central angle $2\pi(1-\sqrt{\frac{2}{3}})$.

49. Into a full conical tumbler of water of depth a and generating angle ϕ a small spherical ball is slowly dropped. Show that in order to cause the greatest possible amount of overflow the radius of the ball should be

$$\frac{a \sin \phi}{\sin \phi + \cos 2\phi}. \quad \text{HINT. See Ex. 17, page 271.}$$

50. It is shown in physics that the intensity of the illumination I received at a point P from a small source of light L varies inversely as the square of the distance PL; that is, $I = \frac{k}{PL^2}$, where k is a constant whose value is to be regarded as the intensity of the illumination at a unit's distance from P. Hence show that if a is the intensity of one light source L at a unit's distance from it and b is similarly the intensity of another source L' at a unit's distance from it, then the point on LL' at which the least total illumination is received from both sources divides the distance LL' in the ratio $\sqrt[3]{a} : \sqrt[3]{b}$.

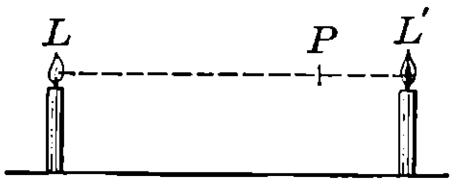


FIG. 49

51. Prove that the problem "to describe a circle with its center on the circumference of a given circle so that the length of the arc intercepted within the given circle shall be a maximum" is reducible to the solving of the trigonometric equation $x = \cot x$.

52. If the portion AB of the tangent to a given curve intercepted by two fixed lines OA, OB is to be a minimum, prove that $PA = NB$, where P is the point of contact and N the foot of the perpendicular let fall on the tangent from O.

53. Show that the greatest rectangle that can be circumscribed about a given rectangle of dimensions a, b is a square the length of whose side is $\frac{1}{\sqrt{2}}(a+b)$.

54. Through a point O on the prolonged diameter AB of a semicircle a secant ORR' is drawn, thus forming an inscribed quadrilateral $ABRR'$. Prove that in case the area of this quadrilateral is a maximum the projection of RR' upon AB is equal to the length of the radius of the circle.

40. Definition. A point, such as P in Fig. 50, at which a curve changes its character from concave upward to concave downward (or *vice versa*) is called a **point of inflection.** Such points are numerous, for example, in the graphs of the equations $y = \sin x$, $y = \cos x$, $y = \tan x$, etc. See Figs. page 370. At a point of inflection the tangent passes through the curve.

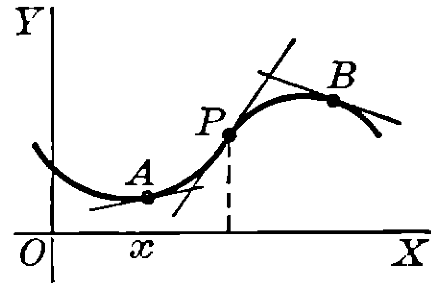


FIG. 50

41. Location of points of inflection. We proceed to establish the following theorem:

Theorem. *Given a function* $f(x)$ *whose second and third derivatives are represented respectively by* $f''(x)$ *and* $f'''(x)$. *If* x_1 *is a special value of* x *such that* $f''(x_1) = 0$ *but* $f'''(x_1) \neq 0$*, *then* x_1 *is the abscissa of a point of inflection upon the curve* $y = f(x)$.

PROOF. Since $f'''(x_1) \neq 0$, let us take first the case in which $f'''(x_1)$ is negative in sign. Then the entire hypotheses become $f''(x_1) = 0$ but $f'''(x_1)$ negative. But these relations imply that the function $f'(x)$ attains a maximum value where $x = x_1$, as follows by applying the theorem of Art. 37 to $f'(x)$. Hence, the slope of the curve $y = f(x)$ attains a maximum value where $x = x_1$. This fact, however, is equivalent to saying that there is a point of inflection where $x = x_1$, for if, as in Fig. 50, the slope of the curve is continually increasing as one selects points A on the curve that are nearer and nearer to P on the left and at the same time the slope is continually diminishing as one selects points B near to P on the right but farther and farther removed from it, then evidently the curve is necessarily concave upward at the left of P and concave downward at the right of P

Similarly, in case $f''(x_1) = 0$ but $f'''(x_1)$ is positive in sign we may show that the curve is concave downward at the left of P and concave upward at the right of P, so that at P there must be a point of inflection.

The proof is thus complete.

Note. The single condition $f''(x_1) = 0$ may be regarded as expressing the *necessary* condition that the curve $y = f(x)$ shall have a point of inflection where $x = x_1$. This single condition, however, is not *sufficient* to assure the existence of such a point, hence the added condition $f'''(x_1) \neq 0$ as stated in the theorem. Compare Note 1, Art. 37. Further consideration of points of inflection, including cases in which $f'''(x_1) = 0$, will be given in Chapter VIII.

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* The symbol $\neq$ is commonly used to denote "is not equal to."

Example 1. Determine the points of inflection of the curve whose equation is

$$y = x^4 - 4x^3 + 8x + 3. \qquad (1)$$

SOLUTION. In the language of the theorem we here have

$$f(x) = x^4 - 4x^3 + 8x + 3$$

and therefore

$$f'(x) = 4x^3 - 12x^2 + 8, \qquad (2)$$

$$f''(x) = 12x^2 - 24x, \qquad (3)$$

$$f'''(x) = 24x - 24. \qquad (4)$$

The values of x for which $f''(x) = 0$ are therefore determined by the equation

$$12x^2 - 24x = 0, \quad \text{or} \quad x(x-2) = 0,$$

the roots of which are $x = 0$ and $x = 2$. Moreover, we see from (4) that $f'''(0) = -24$ and $f'''(2) = +24$, so that, applying the theorem, the curve (1) has points of inflection whose abscissas are $x = 0$ and $x = 2$. To obtain the ordinates of these points, we place $x = 0$ and $x = 2$ in (1), thus obtaining respectively as the corresponding ordinates $y = 3$ and $y = 3$.

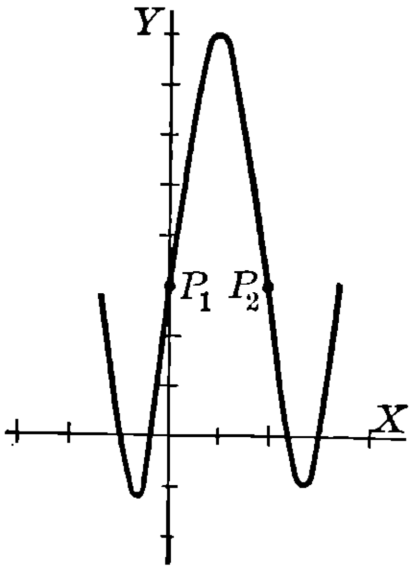

FIG. 51

In summary, the curve (1) has points of inflection at (0, 3) and (2, 3). Moreover, by Note 1 it can have no other such points than these two, since it is only at these that $f''(x) = 0$. Upon drawing the graph of (1) in the usual manner it is found to have the form shown in Fig. 51, the two points of inflection being here denoted by P_1 and P_2.

Example 2. Determine the points of inflection of the curve

$$y = \sin x. \qquad (5)$$

SOLUTION. We have

$$f(x) = \sin x$$

and therefore

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x.$$

Since the roots of the equation $f''(x) = 0$ are in this instance $x = \pm n\pi$, where n is any positive integer, zero included, and since we have $f'''(\pm n\pi) = -\cos(\pm n\pi) = \pm 1$, it follows from the theorem that the curve in question has inflection points whose abscissas are $x = \pm n\pi$. Moreover, upon placing these values of x in (5) we see that the ordinates of all the points just mentioned are equal to zero.

In summary, every point of the type $(\pm n\pi, 0)$, where n is any positive integer, zero included, is an inflection point of the curve (5). Moreover, by Note 1 the curve has no other such points since it is only when $x = \pm n\pi$ that $f''(x) = 0$. The points, unlimited in number, where the curve crosses the X-axis are therefore the inflection points in this case. See page 370.

42. Curve tracing. The usual method of analytic geometry for drawing the graph of a given equation consists in plotting various individual points and then sketching as accurately as possible the smooth curve which passes through them. This method, however, is frequently laborious inasmuch as it is only after a large number of points have been plotted that the essential features of the curve in question become apparent. A much easier method is furnished in most cases by the calculus.

Example 1. Trace the curve

$$y = 4x^3 - 15x^2 + 12x + 1. \tag{1}$$

SOLUTION. Representing the second member of (1) by $f(x)$, we have

$$f(x) = 4x^3 - 15x^2 + 12x + 1$$

and therefore

$$f'(x) = 12x^2 - 30x + 12, \quad f''(x) = 24x - 30, \quad f'''(x) = 24.$$

The equation $f'(x) = 0$ is readily found to have as its roots $x = \frac{1}{2}$ and $x = 2$, and for these we obtain $f''(\frac{1}{2}) = -18$, $f''(2) = +18$. Hence, by the theorem of Art. 37, $x = \frac{1}{2}$ is the abscissa of a maximum point while $x = 2$ is the abscissa of a minimum point. The ordinates of these points, as determined from (1), are found to be $y = 3\frac{3}{4}$ and $y = -3$ respectively. The point $(\frac{1}{2}, 3\frac{3}{4})$ is therefore a maximum point while $(2, -3)$ is a minimum point, and these, by Note 1 of Art. 37, are the only such points possessed by the curve.

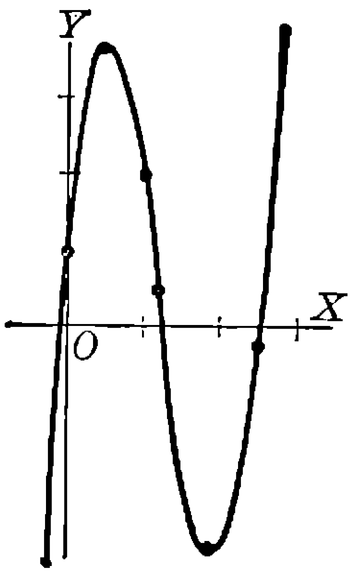

FIG.

Again, the equation $f''(x) = 0$ is seen to have the one root $x = 1\frac{1}{4}$ and for this value of x the corresponding value of y as determined by (1) is found to be $y = \frac{3}{8}$. It follows by the theorem of Art. 41 that $(1\frac{1}{4}, \frac{3}{8})$ is the only point of inflection of the curve.

With this information at hand, the general form of the curve must be as shown in Fig. 52. To add accuracy to the result, however, the special points corresponding to $x = 0$, $x = 1$, $x = 2\frac{1}{2}$ have been plotted to serve as additional guiding points, the ordinates of these points having been found from (1) to be respectively $y = 1$, $y = 2$, $y = -\frac{1}{4}$.

Example 2. Trace the curve $y = \dfrac{x}{x^2 - 1}$. (2)

SOLUTION. It is first desirable to make the following general observations:

(*a*) The point (0, 0) satisfies the equation. Thus, the origin lies on the curve.

(*b*) As x increases from 0 to $+1$ the corresponding values of y are negative. Moreover, they increase indefinitely in numerical value as x approaches the value 1. However, as soon as x passes 1, the values of y are positive and, as x increases indefinitely, they approach zero as a limit.

(c) As x decreases from 0 to -1 the corresponding values of y are positive. Moreover, they increase indefinitely as x approaches the value -1. However, as soon as x passes -1, the values of y are negative and, as x decreases indefinitely in the negative direction, they approach zero as a limit.

These preliminary facts having been noted, we now proceed as in Example 1. Letting $f(x)$ represent the second member of (2), we obtain in the usual manner, after making simple reductions,

$$f'(x) = -\frac{x^2+1}{(x^2-1)^2}, \quad f''(x) = \frac{2x(3+x^2)}{(x^2-1)^3}. \tag{3}$$

The equation $f'(x) = 0$ is therefore equivalent to $x^2 + 1 = 0$. But this equation has no real roots, its roots being imaginary. It follows (see Note 1, Art. 37) that the graph of (2) has no maximum or minimum points. On the other hand, the equation $f''(x) = 0$ is seen from (3) to have but one real root, namely, $x = 0$. Thus, by the Note to the theorem of Art. 41, the origin is the only possible point of inflection which the curve may have.

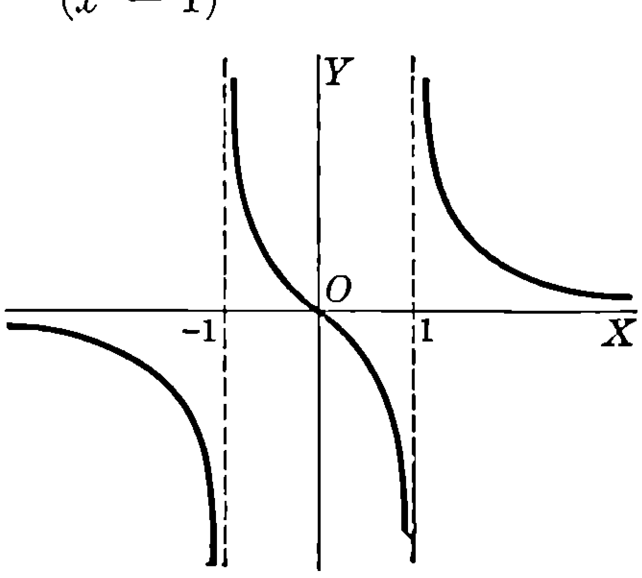

FIG. 53

Upon considering properties (a), (b), (c) together with those just mentioned, it is evident that the curve must have the form shown in Fig. 53.

Note. The subject of curve-tracing will be considered in further detail in Chap. VIII. For a summary of the general properties of a curve that are obtainable directly from its equation, like the properties (a), (b), (c) of Example 2, the student may be referred to the author's text "A Brief Course in Analytic Geometry" (Holt and Co.), pages 23–25.

EXERCISES

Trace each of the following curves, drawing the graph of each:

1. $y = x^2 - x + 2.$

2. $y = \frac{1}{2}x^2 - 3x.$

3. $y = x^2 - 8x + 20.$

4. $y = \frac{1}{3}x^3 + 3x^2 + 8x + 1.$

5. $y = x^3 - 7x + 6.$

6. $y = \dfrac{x}{x-1}.$

7. $y = \dfrac{x^2}{x^2-1}.$

8. $y = \dfrac{x^2}{x-1}.$

9. $y = \dfrac{x}{(x-1)(x+2)}.$

10. $y = \dfrac{(x-2)^3}{(x+2)^2}.$

11. $y = \cos x.$

12. $y = \tan x.$

13. $y = e^x.$

14. $y = \log x.$

15. $y = x^{\frac{1}{x}}.$

16. Prove that no conic has an inflection point.

CHAPTER V

FURTHER APPLICATIONS OF THE DERIVATIVE TO GEOMETRY

43. Introduction. It follows from Art. 9 that if $y = f(x)$ is the equation of any curve, then the slope, m, of the tangent to the curve at any specified point (x_1, y_1) is given by the formula

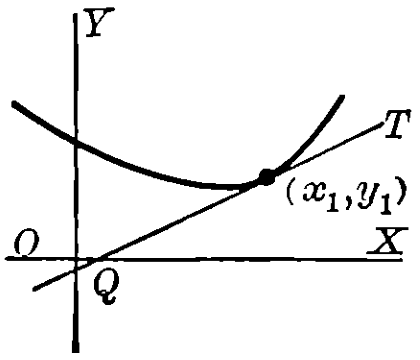

FIG. 54

$$m = \tan \angle XQT = \left[\frac{dy}{dx}\right]_{x=x_1}, \qquad (1)$$

where the symbol $\left[\frac{dy}{dx}\right]_{x=x_1}$ indicates that the derivative $\frac{dy}{dx}$ (which is itself a function of x) is being evaluated for the given special value x_1 of x.*

Thus, the slope of the parabola $y = 4x^2 - 2x + 1$ at the point (2, 13) upon it is equal to 14 since we have in this case

$$m = \left[\frac{dy}{dx}\right]_{x=2} = [8x - 2]_{x=2} = 16 - 2 = 14.$$

More generally, if the given equation, instead of presenting the specific form $y = f(x)$ and thus expressing y *explicitly* as a function of x, is of the type $f(x, y) = 0$, thus including all cases in which y is defined *implicitly* as a function of x (see Art. 34), then the formula corresponding to (1) becomes

$$m = \tan \angle XQT = \left[\frac{dy}{dx}\right]_{\substack{x=x_1 \\ y=y_1}}, \qquad (2)$$

where the symbol $\left[\frac{dy}{dx}\right]_{\substack{x=x_1 \\ y=y_1}}$ indicates that the derivative $\frac{dy}{dx}$ (which will in general contain both x and y) is to be evaluated for the given values x_1 of x and y_1 of y.

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* In this statement and in those that follow it is tacitly assumed that the derivative in question exists; that is, that the limit defined by (5), page 11, has a definite value for the given $f(x)$ when $x = x_1$.

Thus, if the equation of the curve is

$$x^4y^2 - xy^3 + x^2 - 2y = 0 \tag{3}$$

and the given point upon the curve is (0, 0), the slope at this point may be found as follows:

Differentiating (3), member by member, with respect to x (compare Art. 34), we obtain

$$2x^4y\frac{dy}{dx} + 4x^3y^2 - 3xy^2\frac{dy}{dx} - y^3 + 2x - 2\frac{dy}{dx} = 0,$$

or

$$(2x^4y - 3xy^2 - 2)\frac{dy}{dx} = -4x^3y^2 + y^3 - 2x.$$

Hence,

$$\frac{dy}{dx} = \frac{-4x^3y^2 + y^3 - 2x}{2x^4y - 3xy^2 - 2}.$$

At the given point (0, 0) we therefore have

$$\left[\frac{dy}{dx}\right]_{\substack{x=0\\y=0}} = \frac{-0+0-0}{0-0-2} = \frac{0}{-2} = 0,$$

thus indicating that at the point (0, 0) the curve (3) has a slope equal to zero; that is, the direction of the curve at this point is parallel to the X-axis. In this instance the curve itself is not easily drawn.

In order to have a more simple notation, we shall hereafter write the last member of (2) (which contains the second member of (1) as a special case) in the form $\frac{dy_1}{dx_1}$. Then the universal formula for the slope m of any curve $f(x, y) = 0$ at the special point (x_1, y_1) becomes

$$m = \textbf{slope at the point } (x_1, y_1) = \frac{dy_1}{dx_1}.$$

Note. The student should note that the symbol $\frac{dy_1}{dx_1}$ as thus employed indicates the special value which the (variable) expression $\frac{dy}{dx}$ assumes when $x = x_1$ and $y = y_1$.

44. The angle of intersection of two curves. An immediate application of the formula just obtained is in finding the angle at which two given curves intersect. Such an angle, in fact, may always be regarded as the angle between the tangents to the two curves at the point of intersection, and this angle may usually be determined in the manner illustrated in the following example:

Example. Find, correct to 10 minutes, the angle θ of intersection of the two curves (circle and parabola) whose equations are

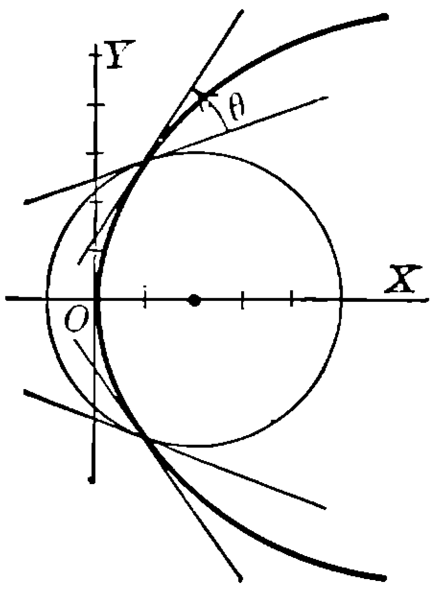

Fig. 55

$$x^2 + y^2 - 4x - 5 = 0, \tag{1}$$

and

$$y^2 - 8x = 0. \tag{2}$$

Solution. It is first necessary to find the points of intersection of the two curves, and for this we solve their equations simultaneously. Thus, substituting for y^2 in (1) the value $8x$ as obtained from (2), we obtain after simple reductions

$$x^2 + 4x - 5 = 0 \tag{3}$$

from which the abscissas of the intersection points of (1) and (2) are to be obtained. Solving (3) by the ordinary methods of solving quadratics, we find its roots to be $x = 1$ and $x = -5$, and placing these values in (2), we find the corresponding values of y to be $y = \pm 2\sqrt{2}$ and $y = \pm 2\sqrt{-10}$ respectively. There are, therefore, two points of intersection of the curves (1) and (2), namely, $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$, it being necessary to discard the additional intersection points $(-5, 2\sqrt{-10})$, $(-5, -2\sqrt{-10})$ because the ordinate of each is imaginary.

Next, the values of the derivative $\dfrac{dy}{dx}$ as obtained in the usual manner for the curves (1) and (2) are respectively

$$\frac{dy}{dx} = \frac{2-x}{y} \quad \text{and} \quad \frac{dy}{dx} = \frac{4}{y}.$$

Hence, by the formula of Art. 43, the slopes m_1 and m_2 of (1) and (2) at the point of intersection $(1, 2\sqrt{2})$ are respectively

$$m_1 = \frac{2-1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{1}{4}\sqrt{2} \quad \text{and} \quad m_2 = \frac{4}{2\sqrt{2}} = \sqrt{2}.$$

Recalling from analytic geometry that the angle θ between the two lines whose slopes are m_1 and m_2 is determined by the formula

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}, \tag{4}$$

it follows that the angle θ between the tangent lines to (1) and (2) at $(1, 2\sqrt{2})$ is determined by the equation

$$\tan\theta = \frac{\sqrt{2} - \frac{1}{4}\sqrt{2}}{1 + \frac{1}{2}} = \frac{\frac{3}{4}\sqrt{2}}{\frac{3}{2}} = \frac{1}{2}\sqrt{2} = \frac{1}{2}(1.41421) = 0.70711 \text{ approximately.}$$

Therefore, by the tables of natural tangents (see page 365)

$$\theta = 35^\circ\, 20' \text{ approximately.}$$

The angle of intersection at $(1, -2\sqrt{2})$ is evidently, for geometric reasons, the same as at $(1, 2\sqrt{2})$. However, this fact can be shown independently if desired by the process above described.

EXERCISES

The following exercises are to be worked by the formula of Art. 43, full use being made of the methods of differentiating implicit functions as there illustrated. Compare with the exercises on page 19.

1. Find the slope of the semicubical parabola $y^2 = x^3$ at (*a*) the point (1, 1); (*b*) the point $(2, 2\sqrt{2})$. See page 368. *Ans.* (*a*) $\frac{3}{2}$; (*b*) $3/\sqrt{2}$.

2. Find the slope of the curve $y^3 = x^2$ at (*a*) the point (1, 1); (*b*) at $(2\sqrt{2}, 2)$.

3. Find the slope of the cissoid $y^2(2\,a - x) = x^3$ at (a, a). See page 369.

4. Show that the tangent to the curve $xy^2 = x + y + 1$ at the point $(2, \frac{3}{2})$ is perpendicular to the line $8\,x - 2\,y = 1$.

5. Show that the tangent to the curve $xy^2 + yx^2 = 2$ at the point (1, 1) makes an angle of 135° with the positive X-axis.

6. Show that, whatever the value of a, the tangent to the curve $x^2y^2 = a^3(x + y)$ at the origin makes an angle of 135° with the positive X-axis.

7. A tangent is drawn to the curve $x^2y = 4\,a^2(2\,a - y)$ (witch of Agnesi) at the point upon it whose abscissa is $a/2$. Show that the slope of this tangent is $-\frac{128}{289}$ and hence determine, correct to 10 minutes, the angle which it makes with the positive X-axis. For the graph of this curve, see page 370.

8. Prove that the tangent to the curve $y^2 \cos x - \sin 3\,x = 0$ at the origin coincides with the Y-axis.

9. Determine the points upon the circle $x^2 + y^2 = 100$ at which the slope of the tangent line is $-\frac{3}{4}$. *Ans.* $(\pm 6, \pm 8)$.

10. Determine the points upon the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (four-cusped hypocycloid) at which the tangents are perpendicular to the line $x - 2\,y + 3 = 0$.

11. Show that the tangents to the folium $x^3 + y^3 = 3\,axy$ at the point where it intersects the parabola $y^2 = ax$ are parallel to the Y-axis. See page 368.

12. Find, to the nearest 10 minutes, the angle of intersection of each pair of curves in the following list. If the curves intersect in more than one point, find the angle of intersection at each.

(*a*) $x - 7\,y + 25 = 0,\ x^2 + y^2 = 25$.
(*b*) $y^2 = 6\,x,\ x^2 + y^2 = 16$.
(*c*) $x^2 + y^2 - 4\,x = 1,\ x^2 + y^2 - 2\,y = 9$.

13. Find, correct to 10 minutes, the angle of intersection of the parabola $x^2 = 4\,ay$ with the witch $x^2y = 4\,a^2(2\,a - y)$. *Ans.* 71° 34′

14. Show that the ellipse $\frac{x^2}{10} + \frac{y^2}{1} = 1$ and the hyperbola $\frac{x^2}{6} - \frac{y^2}{3} = 1$ intersect at right angles at each of four points. Draw a figure.

15. Show that the circle $x^2 + y^2 = 8\,ax$ and the cissoid $y^2(2\,a - x) = x^3$ meet at an angle of 45° at each of two points other than the origin.

16. Prove that the two conics $a_1x^2 + b_1y^2 = 1$ and $a_2x^2 + b_2y^2 = 1$ intersect at right angles only if the following condition is satisfied:

$$\frac{1}{a_1} - \frac{1}{b_1} = \frac{1}{a_2} - \frac{1}{b_2}.$$

45. Equations of tangent and normal. By means of the formula of Art. 43 we may at once establish the following theorem:

Theorem. *The equation of the tangent to the curve $f(x, y) = 0$ at the point (x_1, y_1) is given by the formula*

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1). \tag{1}$$

PROOF. For the proof we have but to recall from analytic geometry that the equation of the straight line passing through a given point (x_1, y_1) and having a given slope m is

$$y - y_1 = m(x - x_1).$$

This equation immediately takes the form stated in the theorem when we use for m its value as stated in the formula of Art. 43. The proof is thus complete.

The foregoing theorem enables one to write also the equation of the normal to a given curve at a given point. In fact, the normal by definition is perpendicular to the tangent, hence by analytic geometry, the slope of the normal is at once obtained by taking the negative reciprocal of the slope of the tangent. To write the normal at the point (x_1, y_1) in any given case, therefore, we have but to write the equation of the line passing through this point and having as its slope the negative reciprocal of the slope of the tangent at this point.

Example. Write the equations of the tangent and the normal to the semi-cubical parabola $y^2 = x^3$ at the point (1, 1).

SOLUTION. We have $y^2 = x^3$, so that, differentiating member by member, we obtain $2\,y\dfrac{dy}{dx} = 3\,x^2$. Hence,

$$\frac{dy}{dx} = \frac{3\,x^2}{2\,y}.$$

At the given point (1, 1) we therefore have

$$\frac{dy_1}{dx_1} = \frac{3}{2}\,\frac{1}{1} = \frac{3}{2}.$$

Hence, by the foregoing theorem, the equation of the tangent at (1, 1) is

$$y - 1 = \tfrac{3}{2}(x - 1),$$

which may be reduced to the form

$$3\,x - 2\,y - 1 = 0.$$

As to the normal at (1, 1), its equation is now immediately written down as

$$y - 1 = -\tfrac{2}{3}(x - 1).$$

This equation may be reduced to the form $2\,x + 3\,y - 5 = 0$.

46. Length of subtangent and subnormal. In the adjoining figure let P_1T be the tangent to the curve AB at P_1 and let P_1N be the normal at the same point. Suppose that P_1T intersects the X-axis at Q and that P_1N intersects the same axis at R; also, suppose that M is the foot of the ordinate drawn at P_1. Then, the length QM is called the **subtangent** of the point P_1 and the length MR is called the **subnormal** of P_1. These two lengths play an important rôle in the properties of certain curves and it is now to be observed that they may be readily formulated in terms of the derivative as follows, it being understood that the equation of the given curve is in the usual form $f(x, y) = 0$.

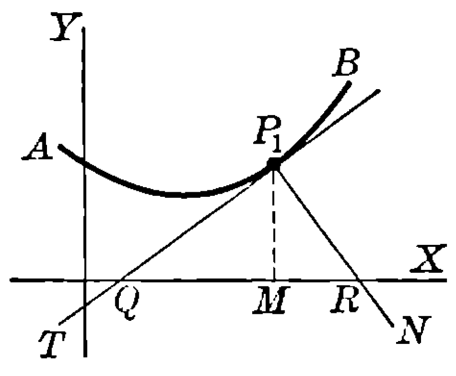

FIG. 56

We have, in fact, from elementary trigonometry together with the formula of Art. 43,

$$QM = \frac{MP_1}{\tan \angle XQP_1} = \frac{y_1}{\frac{dy_1}{dx_1}} = \textit{length of subtangent},$$

$$MR = MP_1 \tan \angle XQP_1 = y_1\frac{dy_1}{dx_1} = \textit{length of subnormal.}$$

Note. As thus formulated, QM will be positive only when Q lies to the left of M, while MR will be positive only when R lies to the right of M, as illustrated in Fig. 56. In all cases, however, the *numerical* values obtained by the formulas represent the lengths of the subtangent and subnormal respectively. The student is advised to confirm this statement by drawing several figures in which Q and R are situated differently as regards M and carrying through the reasoning similar to the foregoing for each.

Example. Find the lengths of the subtangent and the subnormal to the semicubical parabola $y^2 = x^3$ at the point $(2, 2\sqrt{2})$.

SOLUTION. As in the solution of the example in Art. 45, we find that for this curve we have at any point (x, y)

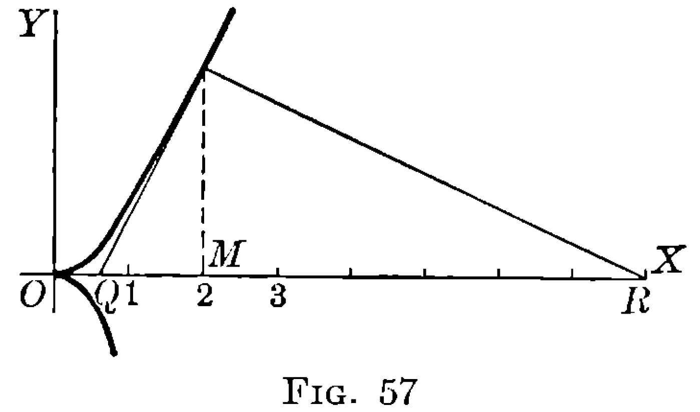

FIG. 57

$$\frac{dy}{dx} = \frac{3\,x^2}{2\,y}.$$

Hence, at $(2, 2\sqrt{2})$ we have

$$\frac{dy_1}{dx_1} = \frac{3\ 2^2}{2\ 2\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

Thus, by use of the foregoing formulas,

$$QM = \textit{Subtangent} = \frac{2\sqrt{2}}{\frac{3}{\sqrt{2}}} = \frac{4}{3}, \quad MR = \textit{Subnormal} = 2\sqrt{2}\ \frac{3}{\sqrt{2}} = 6.$$

EXERCISES

1. Find the equations of the tangent and the normal to the circle $x^2 + y^2 + 2x + 2y - 6 = 0$ at (a) the point $(1, 1)$; (b) the point $(-\frac{7}{5}, -\frac{19}{5})$.

Ans. (a) $x + y = 2, x = y$; (b) $x + 7y = -28, y - 7x = 6$.

2. Find the lengths of the subtangent and the subnormal to the parabola $y^2 = 4x$ at the point $(4, 4)$. *Ans.* 8, 2.

3. Find the equation of the tangent and the normal to each of the following conics at the points indicated. Find also the length of the subtangent and the subnormal in each case:

Conic	*Point*
(a) $x^2 + 2y^2 - 3xy + x - 1 = 0,$	$(1, 1)$;
(b) $9x^2 + 12xy + 4y^2 - 5x + 8y + 2 = 0,$	$(2, -1)$;
(c) $5x^2 + 2xy - y^2 + x - y - 6 = 0,$	$(-1, -1)$.

4. Find the equations of the tangent and the normal to the curve $y^3 - 2y^2 + x^2 = 0$ at the points whose ordinate is 1.

Ans. $2x - y = 1, x + 2y = 3$; $2x + y = -1, 2y - x = 3$.

5. Show that upon the curve $y = \tan x$ the point whose abscissa is $\pi/4$ has its subtangent equal to one fourth its subnormal.

6. Prove that the equation of the tangent at (x_1, y_1) to the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$. Also, state and prove the corresponding result for the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

7. Prove that the equation of the tangent at (x_1, y_1) to the general conic $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ may be written in the form

$$Ax_1x + H(y_1x + x_1y) + By_1y + G(x + x_1) + F(y + y_1) + C = 0.$$

8. Find the equations of the tangent and the normal to the witch $x^2y = 4a^2(2a - y)$ at the point where $x = 2a$. *Ans.* $x + 2y = 4a, 2x - y = 3a$.

9. Find the equations of the tangent and the normal to the curve $x^y = y^x$ at the point $(1, 1)$. *Ans.* $y = x, x + y = 2$.

10. Prove that in the parabola $y^2 = 4px$ (a) the subnormal is the same (constant) for all points; (b) the subtangent is bisected at the vertex.

11. Prove that the subtangent to the curve $y = a^x$ is the same at all points, and is equal to $\frac{1}{\log a}$.

12. Show that in the equilateral hyperbola $2xy = a^2$ the areas of the triangles formed by tangents and the coördinate axes is a constant, and is equal to a^2.

13. Show that the sum of the intercepts of a tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ on the coördinate axes is a constant, and is equal to a.

14. Show that for the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ the portion of any tangent included between the coördinate axes is equal to a and hence is a constant.

15. Show that the curve $y = a^{\frac{x}{c}}$ has a constant subtangent.

47. Definition. In Fig. 58 let P be a fixed point upon the arc AB and let P' be a neighboring point situated at a distance Δs from P as measured along AB. Let the tangents at P and P' make the angles τ and $\tau + \Delta\tau$ respectively with the positive X-axis, thus causing $\Delta\tau$ to represent the angle between these two tangents. Then $\Delta\tau$ evidently measures the entire change of direction of the curve as one passes along it from P to P', while the ratio $\frac{\Delta\tau}{\Delta s}$ comes to represent the *average* change of direction per unit of arc as one passes from P to P' * If P' be now allowed to approach P, thus making both Δs and $\Delta\tau$ approach zero, this ratio $\frac{\Delta\tau}{\Delta s}$ will approach (in general) a limit which, by Art. 9, will be equal to the derivative $\frac{d\tau}{ds}$ evaluated at P This limit pertains specifically to the point P and may be considered as measuring the rate of change of direction of the curve *at* this point. The numerical value of this derivative is called the **curvature** † of the curve at P.

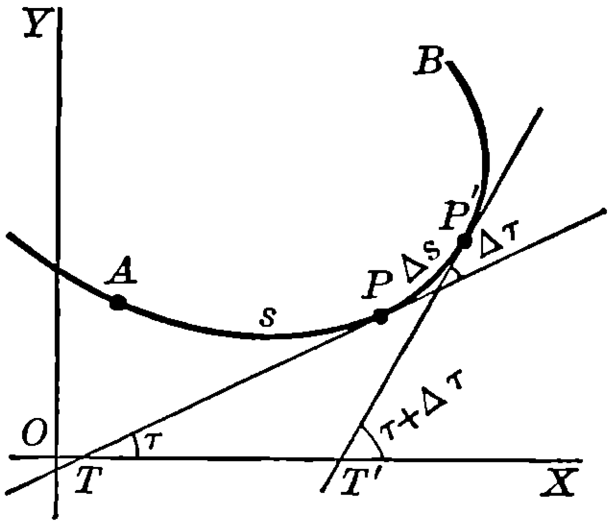

FIG. 58

In summary, then, we have by definition

$$\textit{Curvature at } P = \left|\frac{d\tau_1}{ds_1}\right|, \tag{1}$$

where the symbol $\left|\frac{d\tau_1}{ds_1}\right|$ indicates the numerical value of the derivative $\frac{d\tau}{ds}$ when evaluated for the special values of s and τ that pertain to the point P.

---

* The change in direction corresponding to a unit's distance along PP' will evidently vary according to where the unit is laid off, but if the *whole* change of direction $\Delta\tau$ is divided by the *whole* distance Δs, the result will be the *average* change of direction per unit of arc.

† It follows from Art. 9 that $\frac{d\tau}{ds}$ will be a positive or negative number according as the angle τ is increasing or decreasing as s passes by increasing values through the particular value which it has at P. However, the *numerical value* of $\frac{d\tau}{ds}$ is, of course, always positive.

Note. The conception of "curvature" as thus accurately formulated is readily seen to be in accord with the commonly accepted use of the word. In fact, as ordinarily understood, the curvature at any point of a curve is great or small according as there is a great or small change in the direction of the curve as one passes through the point, and it is this idea that is basal in the foregoing accurate definition. The student will find as he proceeds in the study of mathematics that the notion of curvature plays an important part both in geometry and mechanics.

48. Curvature of the circle. It is important to observe that in case the arc AB (Fig. 58) is that of a circle, the value of the curvature at every point P is the same. That this is a fact might be inferred directly upon noting that as one passes along a circle the change in direction corresponding to any given length of arc passed over is everywhere the same. The fact may, however, be more accurately and completely established in the form of a theorem as follows, the proof being made to depend upon the fundamental formula just mentioned.

Theorem. *The curvature of a circle is the same at every point, and is equal to* $\frac{1}{r}$, *where* r *is the radius.*

Proof. For convenience, place the circle so that its center is on the Y-axis at a distance from the origin, O, equal to the radius. Let the arc s be measured from O and let θ be the central angle, measured in radians, corresponding to s. Then, if r is the radius, we have by elementary geometry

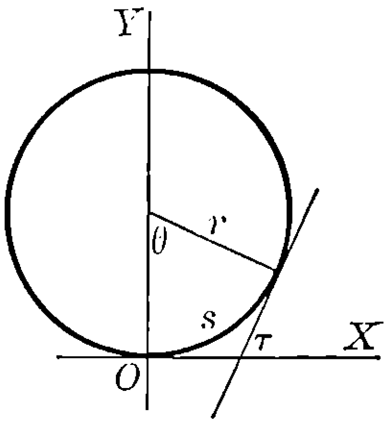

Fig. 59

$$s = r\theta. \tag{1}$$

But inasmuch as the sides of the angle τ are respectively perpendicular to those of the angle θ, we have in the present case $\theta = \tau$. Thus (1) takes the form

$$s = r\tau \qquad \text{or} \qquad \tau = \frac{1}{r}s, \tag{2}$$

from which we obtain at once

$$\frac{d\tau}{ds} = \frac{1}{r}.$$

Therefore the curvature, as determined by the general formula of Art. 47, is the same for all values of s and τ, being equal to the constant $\frac{1}{r}$

The proof is thus complete.

49. Curvature in rectangular coördinates. In order to use the general formula of Art. 47 to determine the curvature of a given curve at a given point it is evidently necessary that the angle τ be known in advance as a function of the arc length s, as illustrated in equation (2) of Art. 48. To obtain this functional relation, however, for a curve whose equation is given in the usual form $f(x, y) = 0$ is frequently difficult. It therefore becomes desirable, if possible, to convert the formula into a more convenient form. This we shall do presently, but it is desirable first to establish the following lemma:

Lemma. *The variable distance s measured along any given curve $y = f(x)$ from some fixed starting point, as A, to a variable point $P = (x, y)$ on the curve is a function of x such that*

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{1}$$

Similarly s, when regarded as a function of y, is such that

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \tag{2}$$

PROOF. Let any point $P' = (x + \Delta x, y + \Delta y)$ be chosen on the curve near to P Let the distance from P to P' as measured along the curve be Δs and let the straight-line distance (chord) connecting P to P' be represented by $\overline{PP'}$ Then we shall evidently have

$$\overline{PP'}^2 = (\Delta x)^2 + (\Delta y)^2.$$

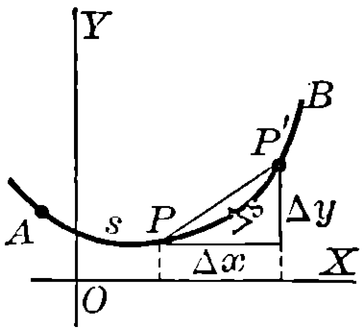

FIG. 60

By dividing both members of this equation by $(\Delta x)^2$ and subsequently writing the left member of the result in a form wherein $(\Delta s)^2$ enters in such a manner as to cancel from numerator and denominator, we obtain

$$\left(\frac{\overline{PP'}}{\Delta s}\right)\left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

If we now allow Δx, and hence Δy and Δs, to approach zero, noting that $\lim\limits_{\Delta s \to 0} \frac{\overline{PP'}}{\Delta s} = 1$, we obtain

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

Hence

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

as was to be proved in (1).

The proof of (2), being similar to the above, is left to the student.

The foregoing lemma having been established, we may now establish the following theorem whereby the formula for curvature takes a form more adaptable to general use than (1) of Art. 47:

Theorem. *The curvature K of the curve $f(x, y) = 0$ at the point (x_1, y_1) is determined by the formula*

$$K = \textit{abs. val.} \frac{\dfrac{d^2y_1}{dx_1^2}}{\left[1 + \left(\dfrac{dy_1}{dx_1}\right)^2\right]^{\frac{3}{2}}},$$

where $\dfrac{dy_1}{dx_1}$ and $\dfrac{d^2y_1}{dx_1^2}$ indicate respectively the values assumed by $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ when $x = x_1$ and $y = y_1$ and where abs. val. indicates that the absolute, or numerical, value only of the expression is to be taken.

PROOF. Returning to Fig. 58, let us suppose that the coördinates of P and P' are respectively (x, y) and $(x + \Delta x, y + \Delta y)$ and let us now write the fundamental ratio $\dfrac{\Delta\tau}{\Delta s}$ in the form

$$\frac{\Delta\tau}{\Delta s} = \frac{\dfrac{\Delta\tau}{\Delta x}}{\dfrac{\Delta s}{\Delta x}}.$$

Then, using Theorem III of Art. 5, we may write

$$\frac{d\tau}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\tau}{\Delta s} = \frac{\lim\limits_{\Delta x = 0} \dfrac{\Delta\tau}{\Delta x}}{\lim\limits_{\Delta x = 0} \dfrac{\Delta s}{\Delta x}}. \tag{1}$$

We proceed to evaluate the two limits here appearing in the last member.

By Art. 8 we have at once $\lim\limits_{\Delta x \to 0} \dfrac{\Delta\tau}{\Delta x} = \dfrac{d\tau}{dx}$. But $\tan\tau = \dfrac{dy}{dx}$, or $\tau = \text{arc} \tan \dfrac{dy}{dx}$. Hence, using the theorem of Art. 30, we obtain

as the numerator of (1)

$$\frac{d\tau}{dx} = \frac{\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)}{1+\left(\dfrac{dy}{dx}\right)^2} = \frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}. \tag{2}$$

As to the denominator of (1), we have by the foregoing lemma

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{3}$$

We have now but to substitute the forms (2) and (3) in (1) to obtain

$$\frac{d\tau}{ds} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{4}$$

Hence the curvature, which by the definition in Art. 47 is the absolute, or numerical, value of $\dfrac{d\tau}{ds}$, will be equal to the absolute value of the expression here appearing on the right when evaluated at the point P in question.

The proof is thus complete.

Note. It may be observed that the expression (4) whose absolute value is the curvature K, is positive or negative according as its numerator, $\dfrac{d^2y}{dx^2}$, is positive or negative and hence, by the reasoning in Art. 41, according as the curve is concave upward or concave downward at the point in question. This assumes however, that $\dfrac{d^3y}{dx^3} \neq 0$ at this point.

Example. Find the curvature of the parabola $y^2 = 8x$ at the point $(2, -4)$.

SOLUTION. We here have

$$\frac{dy}{dx} = \frac{4}{y};\ \frac{d^2y}{dx^2} = -\frac{4}{y^2}\frac{dy}{dx} = -\frac{16}{y^3}.$$

Substituting in the formula of the foregoing theorem, we obtain as the desired curvature at the point $(2, -4)$

$$K = \frac{-\dfrac{16}{(-4)^3}}{\left[1+\left(\dfrac{4}{-4}\right)^2\right]^{\frac{3}{2}}} = \frac{1}{4\cdot 2^{\frac{3}{2}}} = \frac{\sqrt{2}}{16}.$$

50. Circle and radius of curvature. Let P be a given point upon an arc AB and let P' and P'' be two points neighboring to P and also upon AB, as illustrated in Fig. 61(a). Draw the circle determined by the three points P, P', P'' and suppose that C' and R' represent respectively the center and the radius of this circle. Such a circle will, in general, pass through AB at each of the points P, P', P'' Now suppose that the same construction is repeated, P remaining the same in position as before but with P' and P'' chosen nearer to P than before. The resulting circle will evidently have a different center C' and radius R' than before. Again, suppose the same construction is repeated but with P' and P'' still nearer to the fixed point P. A still different circle will result. Suppose that this process is now continued indefinitely, the points P' and P'' thus approaching P as a limit. The resulting family of circles, indefinite in number, will evidently approach in general a limiting circle which is tangent to AB at P, as illustrated in Fig. 61(b). This limiting circle is called the **circle of curvature** of AB at P; its center C is called the **center of curvature,** and its radius R is called the **radius of curvature** of AB at P Moreover, it is possible to show, though we shall for brevity omit the proof, that the curvature of this circle at P is the same as the curvature K of the arc itself at this point. In other words, recalling the theorem of Art. 48, we have the equation

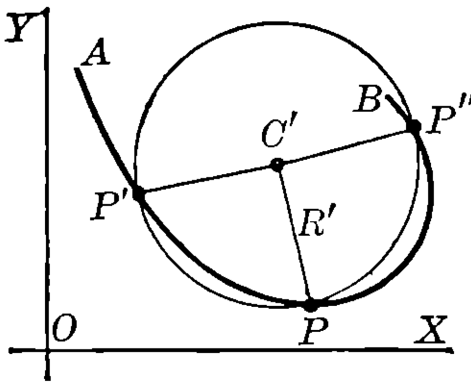

FIG. 61 (a)

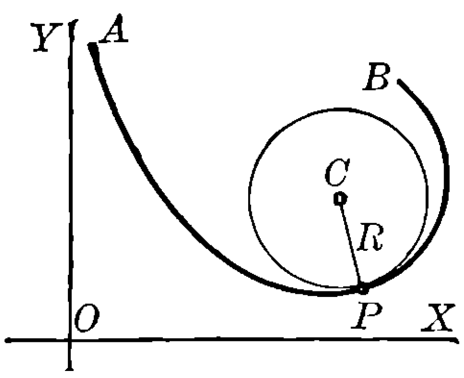

FIG. 61 (b)

$$K = \frac{1}{R}. \tag{1}$$

Solving (1) for R, we thus have the following fundamental theorem:

Theorem. *The radius of curvature R of any arc AB at any point P is given by the formula*

$$R = \frac{1}{K},$$

where K may be determined by the theorem of Art. 49.

Note. The radius of curvature is a magnitude of fundamental importance, particularly in dynamics; that is, in the general study of motion. For example, it can be shown that if a small particle, such as a bead, moves along any arc AB, the centrifugal force which it develops at any point varies inversely as the radius of curvature of the arc at that point.

Example. Determine the radius of curvature of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at the extremity of the major axis.

SOLUTION. We first observe from analytic geometry that this ellipse has its center at the origin, that its semiaxes are 4 and 3, and that the major axis lies along the Y-axis. Thus, the coördinates of the upper extremity of the major axis are (0, 4), so that the problem is that of determining the radius of curvature at this point.

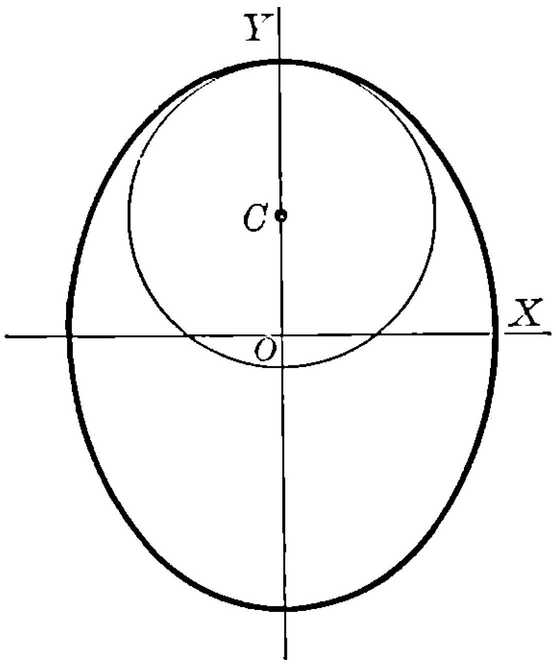

FIG. 62

Writing the equation in the form

$$16\,x^2 + 9\,y^2 = 144$$

and differentiating, we obtain

$$32\,x + 18\,y\frac{dy}{dx} = 0, \tag{2}$$

and hence, when $x = 0$, $y = 4$, we have $\frac{dy}{dx} = 0$.

Upon differentiating (2) we obtain

$$32 + 18\,y\frac{d^2y}{dx^2} + 18\left(\frac{dy}{dx}\right)^2 = 0, \tag{3}$$

and hence, recalling that $y = 4$ and $\frac{dy}{dx} = 0$ at the point (0, 4) in question, we find that at the same point the value of $\frac{d^2y}{dx^2}$ is $-\frac{4}{9}$.

The value of K at the point is therefore, by Art. 49,

$$K = abs.\ val.\ \frac{-\frac{4}{9}}{[1+0]^{\frac{3}{2}}} = abs.\ val.\left(-\frac{4}{9}\right) = \frac{4}{9}.$$

The desired radius of curvature at (0, 4) is therefore, by the foregoing theorem,

$$R = \frac{1}{\frac{4}{9}} = \frac{9}{4} = 2\tfrac{1}{4}.$$

Note. If in the foregoing example it had been desired to determine R at the extremity of the *minor* axis; that is, at the point (3, 0), the method would have to be altered inasmuch as the value of $\frac{dy}{dx}$ at this point is infinite. However, this difficulty is easily removed by first interchanging the X- and Y-axes. Then the equation of the ellipse becomes

$$\frac{x^2}{16} + \frac{y^2}{9} = 1,$$

and the extremity in question becomes (0, 3). Reasoning as before, we find that at this point $R = 5\frac{1}{3}$. The student is advised to carry through the steps.

EXERCISES

1. Show that the curvature of any straight line is equal to zero at every point and draw such inferences as you can regarding its radius of curvature.

HINT. Take the equation of the straight line in the form $y = mx + b$, where m and b are constants.

2. Find the radius of curvature of the hyperbola $xy = 3$ at $(2, 1\frac{1}{2})$. *Ans.* $\frac{125}{48}$.

3. Find the radius of curvature of the semicubical parabola $y^2 = x^3$ at the point (1, 1). *Ans.* $\dfrac{6}{13\sqrt{13}}$.

4. Show that the curvature of the curve $y = \sin x$ at any one of its highest or lowest points is equal to 1, and prove that the curve $y = \cos x$ has the same property. See Fig. 370.

5. Determine the radius of curvature of each of the following curves at the point indicated for each:

Curve	*Point*
(*a*) *Parabola*, $y = 4x^2 + 2x + 1$,	$(-1, 3)$;
(*b*) *Ellipse*, $4x^2 + 9y^2 = 100$,	$(4, 2)$;
(*c*) *Hyperbola*, $4x^2 - 9y^2 = 100$,	$(6, \frac{2}{3}\sqrt{11})$;
(*d*) *Cissoid*, $y^2(2a - x) = x^3$,	(a, a);
(*e*) *Logarithmic Curve*, $y = \log x$,	$(1, 0)$.

6. Show that at a point of inflection upon any curve the curvature is zero.

7. Show that the radius of curvature at any point (x, y) upon the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is equal to $\dfrac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$ and hence show that the radii of curvature at the ends of the major and minor axes are respectively b^2/a and a^2/b.

8. Show that the radius of curvature of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ at any point (x, y) is equal to y^2/a. For the graph of this curve, see page 368.

9. For each of the following curves, show that the radius of curvature at any point (x, y) may be expressed in the form indicated:

(*a*) *Cubical Parabola*, $y = ax^3$, $R = \dfrac{(1 + 9a^2x^4)^{\frac{3}{2}}}{6ax}$;

(*b*) *Semicubical Parabola*, $y^2 = ax^3$, $R = \dfrac{x^{\frac{1}{2}}(4 + 9ax)^{\frac{3}{2}}}{6a^{\frac{1}{2}}}$;

(*c*) *Parabola*, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $R = \dfrac{2(x + y)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$;

(*d*) *Hypocycloid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $R = 3(axy)^{\frac{1}{3}}$.

10. If a bead slides with uniform velocity around the ellipse $4x^2 + 9y^2 = 100$, compare the centrifugal force which it has when at the extremity of the major axis with that which it has at the extremity of the minor axis.

HINT. See Note, Art. 50; also Ex. 7 above.

51. Parametric equations. Instead of using a single equation between the coördinates x and y to describe a curve it is frequently an advantage to use two equations expressing x and y in terms of a third variable, or **parameter,** which we may represent by α. Such a pair of equations have the type forms

$$\begin{cases} x = f(\alpha), \\ y = \phi(\alpha), \end{cases} \tag{1}$$

and these are called the **parametric equations** of the curve corresponding to the parameter α.

Thus, instead of the single equation

$$x^2 + y^2 = r^2 \tag{2}$$

used to describe all points upon the circle whose center is at the origin and whose radius is r, we may use the parametric equations

$$\begin{cases} x = r \cos \alpha, \\ y = r \sin \alpha, \end{cases} \tag{3}$$

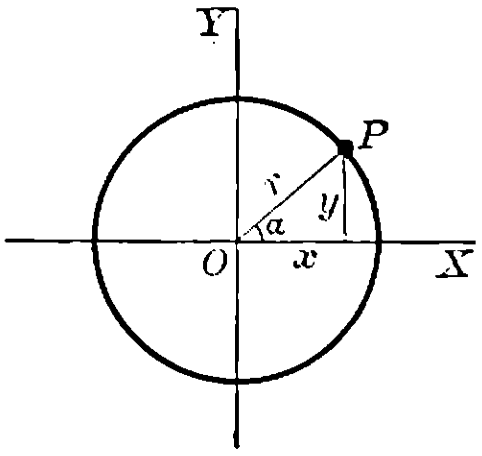

FIG. 63

the parameter α here being the central angle corresponding to any point (x, y) upon the circle. In fact, Fig. 63 makes it clear geometrically that the values of x, y, and α pertaining to any point P upon the circle are connected by means of the relations (3). Moreover, we can actually obtain (2) from (3) by eliminating α from (3) as follows:

Squaring each of the equations (3) and adding the resulting equations, we have

$$x^2 + y^2 = r^2(\sin^2 \alpha + \cos^2 \alpha),$$

and this reduces to (2) when it is recalled from trigonometry that, whatever the angle α, we have $\sin^2 \alpha + \cos^2 \alpha = 1$.

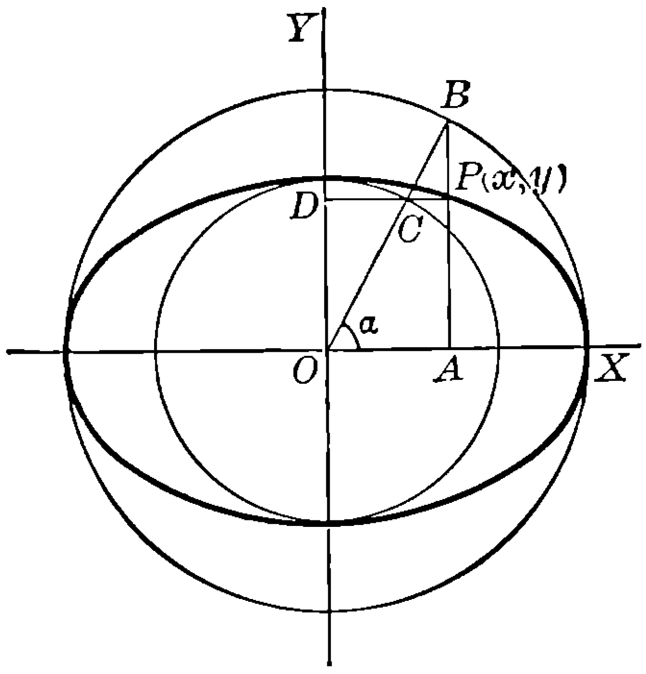

FIG. 64

Similarly, it is shown in analytic geometry that the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may have its equation expressed in the parametric form

$$\begin{cases} x = a \cos \alpha, \\ y = b \sin \alpha, \end{cases}$$

the parameter α in this case being the so-called *eccentric angle* corresponding to the point (x, y).

For the purposes of the calculus it is desirable to state and prove the following theorem relative to the forms which the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ take when x and y are joined together through parametric equations; that is, through two equations of the type (1).

Theorem. *If x and y are functionally related through the parametric equations*

$$x = f(\alpha), \quad y = \phi(\alpha),$$

then the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ corresponding to any assigned value of α are given respectively by the formulas

$$\frac{dy}{dx} = \frac{\phi'(\alpha)}{f'(\alpha)}, \qquad \frac{d^2y}{dx^2} = \frac{f'(\alpha)\phi''(\alpha) - \phi'(\alpha)f''(\alpha)}{[f'(\alpha)]^3},$$

where the primes denote derivatives with respect to α.

PROOF. By the theorem of Art. 16 together with the lemma of Art. 25, we may write

$$\frac{dy}{dx} = \frac{dy}{d\alpha} \cdot \frac{d\alpha}{dx} = \frac{\frac{dy}{d\alpha}}{\frac{dx}{d\alpha}} = \frac{\phi'(\alpha)}{f'(\alpha)}. \tag{4}$$

Thus, the first part of the theorem is established.

Again, we may write for similar reasons

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\alpha}\left(\frac{dy}{dx}\right) \cdot \frac{d\alpha}{dx} = \frac{\frac{d}{d\alpha}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\alpha}} = \frac{\frac{d}{d\alpha}\left(\frac{dy}{dx}\right)}{f'(\alpha)}. \tag{5}$$

But, upon differentiating (4) with respect to α, making use of Theorem III of Art. 15, we have

$$\frac{d}{d\alpha}\left(\frac{dy}{dx}\right) = \frac{d}{d\alpha}\left(\frac{\frac{dy}{d\alpha}}{\frac{dx}{d\alpha}}\right) = \frac{\frac{dx}{d\alpha}\frac{d^2y}{d\alpha^2} - \frac{dy}{d\alpha}\frac{d^2x}{d\alpha^2}}{\left(\frac{dx}{d\alpha}\right)^2} = \frac{f'(\alpha)\phi''(\alpha) - \phi'(\alpha)f''(\alpha)}{[f'(\alpha)]^2}.$$

When this expression for $\frac{d}{d\alpha}\left(\frac{dy}{dx}\right)$ is substituted in (5), the result obtained for $\frac{d^2y}{dx^2}$ is seen to be as stated in the theorem, thus finishing the proof.

Example. Determine the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point upon the curve

$$x = \alpha^2 + 2, \qquad y = 3\,\alpha^3 + \alpha \tag{6}$$

corresponding to the value $\alpha = 1$.

SOLUTION. In the language of the theorem we here have $f(\alpha) = \alpha^2 + 2$ and $\phi(\alpha) = 3\,\alpha^3 + \alpha$. Hence $f'(\alpha) = 2\,\alpha$, $f''(\alpha) = 2$, and $\phi'(\alpha) = 9\,\alpha^2 + 1$, $\phi''(\alpha) = 18\,\alpha$. The theorem thus gives

$$\frac{dy}{dx} = \frac{9\,\alpha^2 + 1}{2\,\alpha}, \qquad \frac{d^2y}{dx^2} = \frac{2\,\alpha \quad 18\,\alpha - (9\,\alpha^2 + 1) \quad 2}{8\,\alpha^3} = \frac{9\,\alpha^2 - 1}{4\,\alpha^3}.$$

For the given value $\alpha = 1$ it thus appears that $\frac{dy}{dx} = 5$ and $\frac{d^2y}{dx^2} = 2$.

The student is advised to obtain a notion of the geometric form of the curve (6) by plotting carefully a variety of points upon its graph. In order to do this, assign various values to α, noting for each the point (x, y) determined by the two equations.

52. Applications. The determination of the subtangent, subnormal, and other important geometric magnitudes for curves whose equations are expressed in parametric form is readily carried out, as illustrated in the following example relative to the cycloid.*

---

* The path described by a point upon the circumference of a circle as the circle rolls without sliding upon a fixed straight line is called a **cycloid**. If the radius of the rolling circle is a and if P is the generating point and M the point of contact of the circle with the fixed line, then in all positions of P the arc PM will be equal to the line-segment OM, where O is regarded as the initial position of P, it being assumed that O is on the given fixed line. If, therefore, we take O as origin and the fixed line as X-axis and represent the (variable) angle PCM by θ, we have for all positions of P

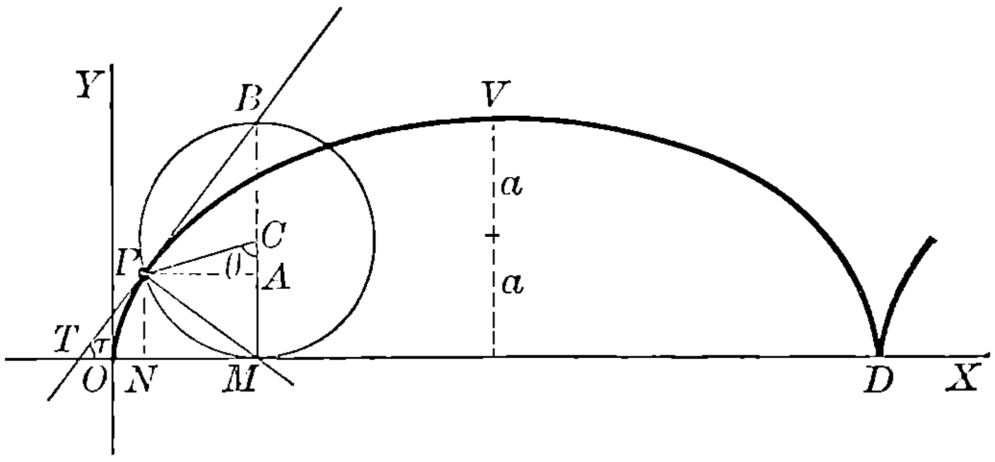

FIG. 65. THE C

$$x = OM - NM = a\theta - a\sin\theta = a(\theta - \sin\theta),$$
$$y = NP = MC - AC = a - a\cos\theta = a(1 - \cos\theta).$$

These, then, may be taken as the parametric equations of the cycloid, the central angle θ being the parameter.

By eliminating θ from the foregoing equations it may be shown that the equation of the cycloid in rectangular coördinates is

$$x = a \operatorname{arc}\cos\left(\frac{a - y}{a}\right) - \sqrt{2\,ay - y^2}.$$

Example. It being given that the parametric equations of the cycloid are

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta), \tag{1}$$

find the lengths of the subtangent and the subnormal and the value of the radius of curvature at the point on the cycloid for which $\theta = \frac{\pi}{2}$.

SOLUTION. We have

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta, \tag{2}$$

and therefore, by the theorem of Art. 51, we may write

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}.$$

The value of $\frac{dy}{dx}$ at the given point, namely, the point for which $\theta = \frac{\pi}{2}$, is therefore

$$\frac{dy_1}{dx_1} = \frac{\sin\frac{\pi}{2}}{1 - \cos\frac{\pi}{2}} = 1. \tag{3}$$

Moreover, the values of x and y at the point in question are, from (1),

$$x_1 = a\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) = \frac{\pi a}{2} - a; \qquad y_1 = a\left(1 - \cos\frac{\pi}{2}\right) = a. \tag{4}$$

Substituting the values given by (3) and (4) in the formulas of Art. 46, we obtain

$$\textit{Length of subtangent} = \frac{y_1}{\frac{dy_1}{dx_1}} = \frac{a}{1} = a,$$

$$\textit{Length of subnormal} = y_1\frac{dy_1}{dx_1} = a \cdot 1 = a.$$

Again, from (2) we obtain

$$\frac{d^2x}{d\theta^2} = a\sin\theta, \qquad \frac{d^2y}{d\theta^2} = a\cos\theta,$$

and therefore, by use of the theorem of Art. 51, we may write

$$\frac{d^2y}{dx^2} = \frac{a(1 - \cos\theta)\ a\cos\theta - a\sin\theta\ a\sin\theta}{a^3(1 - \cos\theta)^3}.$$

The value of $\frac{d^2y}{dx^2}$ at the given point is therefore

$$\frac{d^2y_1}{dx_1^2} = \frac{a^2\left(1 - \cos\frac{\pi}{2}\right)\cos\frac{\pi}{2} - a^2\sin^2\frac{\pi}{2}}{a^3\left(1 - \cos\frac{\pi}{2}\right)^3} = -\frac{1}{a}. \tag{5}$$

Substituting the values given in (3) and (5) in the formula of Art. 49, we obtain

$$K = \textit{curvature} = \textit{abs. val.}\ \frac{-\frac{1}{a}}{[1 + 1^2]^{\frac{3}{2}}} = \frac{1}{2\,a\sqrt{2}}.$$

Hence, by the theorem of Art. 50, we have at the point in question

$$\textit{Radius of curvature} = \frac{1}{K} = 2\,a\sqrt{2}.$$

EXERCISES

1. Having given the curve whose parametric equations are

$$x = (\alpha + 1)^3,\ y = (\alpha - 1)^3,$$

find (*a*) the coördinates of the point corresponding to the parameter value $\alpha = 0$; (*b*) the slope of the curve at this point. *Ans.* (1, − 1); 1.

2. Obtain the equation of the tangent to the curve mentioned in Ex. 1 at the point whose parameter value is $\alpha = 2$. *Ans.* $x - 9\,y = 18$.

3. Show that the two curves

$$\begin{cases} x = \alpha(\alpha + 1), \\ y = \alpha(\alpha - 1), \end{cases} \qquad \begin{cases} x = \alpha(\alpha^2 + 3\,\alpha + 3), \\ y = 3\,\alpha(\alpha + 1) \end{cases}$$

intersect each other at right angles at the point for which $\alpha = 0$; that is, at the origin (0, 0).

4. Find, in terms of the parameter t, the values of the subtangent, the subnormal, and the radius of curvature at any point upon each of the following curves:

$$(a)\ \begin{cases} x = 3\,t^2, \\ y = 3\,t - t^3; \end{cases} \qquad (b)\ \begin{cases} x = t^2 + t, \\ y = e^t; \end{cases} \qquad (c)\ \begin{cases} x = \sin t, \\ y = \cos t. \end{cases}$$

5. Express the values of the subtangent, the subnormal and the radius of curvature at any point upon each of the following curves:

(*a*) The four-cusped hypocycloid $\begin{cases} x = a\cos^3\theta, \\ y = a\sin^3\theta; \end{cases}$

(*b*) The four-cusped epicycloid $\begin{cases} x = \frac{5}{4}\,a\cos\theta - \frac{1}{4}\,a\cos 5\,\theta, \\ y = \frac{5}{4}\,a\sin\theta - \frac{1}{4}\,a\sin 5\,\theta; \end{cases}$

(*c*) The cardioid $\begin{cases} x = a(2\cos\theta - \cos 2\,\theta), \\ y = a\,(2\sin\theta - \sin 2\,\theta). \end{cases}$

6. Show that at any point on the cycloid the radius of curvature is equal to twice the length of the normal.

7. In Fig. 65 suppose that the circle is rolling in the direction OX, the center moving at the rate of 2 inches per second. How fast will the point P be rising when (*a*) $\theta = 30°$; (*b*) $\theta = 45°$; (*c*) $\theta = 60°$? Investigate also for $\theta = 90°$, 180°, 210°, 300°.

CHAPTER VI

RATES

53. Introduction. It was shown in Art. 11 that whenever a point moves along a given curve $y = f(x)$ in the general direction from left to right (more exactly, in such a way that x is continually increasing) the value of the derivative $\frac{dy}{dx}$ at any fixed point $P = (x, y)$ on the curve is equal to the ratio of the rate of change of y to the rate of change of x at the point P in question. Moreover, it was shown that y will be increasing or decreasing at P according as the sign of $\frac{dy}{dx}$ at this point is positive or negative.

In the present chapter we shall show in further detail the connection between the derivative and the idea of rates.

54. Velocity and acceleration in rectilinear motion. If a point P moves along a straight line in such a way that equal distances are everywhere passed over in equal times, the motion is said to be **uniform.** In such motion the ratio of the distance s to the time t required to pass over it is everywhere the same and is called the **velocity** v of the moving point; that is, in uniform rectilinear motion we have the law

$$v = \frac{s}{t}. \tag{1}$$

If on the other hand, the motion, though confined to a straight line, is not uniform, we may proceed as follows to define the notion of velocity.

Suppose that at the instant when the moving point arrives at a given position, as P, the time that has elapsed in order to reach this point is t while the space that has been passed over is s. Moreover, suppose that during a small interval of time, Δt, immediately following t the moving point passes over the additional distance Δs. Then it follows from (1) that, at least if Δt and hence Δs are very small, the ratio $\frac{\Delta s}{\Delta t}$ will form a good approximation to the

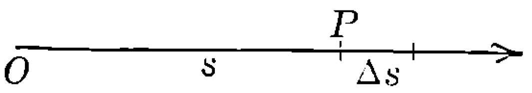

FIG. 66

velocity of the moving point while traversing the distance Δs. Since the correctness of this approximation grows better and better as Δt is taken smaller and smaller, it is natural to define the velocity *at P* as the value of

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

But this limit, by Art. 9, is equal to the value of the derivative $\frac{ds}{dt}$ at P In summary, then, we have the following law pertaining to all rectilinear motion:

$$\boldsymbol{v = velocity\ at\ P = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt},} \tag{2}$$

it being understood that the derivative $\frac{ds}{dt}$ is to be evaluated at the point P in question.

Example 1. A point moves along a straight line in such a manner that the distance passed over varies as the cube of the time. If the point is observed to be 3 feet from the starting point at the end of 2 seconds, what will be its distance from the starting point and its velocity at the end of 6 seconds?

SOLUTION. The general law of the motion when stated in the form of an equation becomes

$$s = kt^3, \tag{3}$$

where k is some constant. Moreover, by hypothesis $s = 3$ when $t = 2$. Hence (3) gives $3 = k \cdot 8$ so that $k = \frac{3}{8}$. The precise law of the motion is therefore

$$s = \tfrac{3}{8} t^3. \tag{4}$$

The desired distance from the starting point at the end of 6 seconds will therefore be

$$s_1 = \tfrac{3}{8} \cdot 6^3 = 81 \text{ feet.}$$

As to the velocity, we have from (4)

$$\frac{ds}{dt} = \frac{9}{8} t^2, \tag{5}$$

so that, in accordance with the explanations of this article, the desired velocity at the end of 6 seconds will be

$$v_1 = \tfrac{9}{8} \cdot 6^2 = 40\tfrac{1}{2} \text{ feet per second.}$$

The rate of change of the velocity at any point P is called the **acceleration** at that point and is commonly represented by the letter j. In the case of uniform rectilinear motion the acceleration is equal to zero at every point since there is no change whatever in the velocity, while in the case of non-uniform rectilinear motion we have but to reason as in the foregoing study of the velocity v in order to arrive at the following universal law.

$$j = \textit{acceleration at } P = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2s}{dt^2}, \qquad (6)$$

it being understood that the derivative $\frac{dv}{dt}$, or $\frac{d^2s}{dt^2}$, is to be evaluated at the point P in question.

Example 2. Find the acceleration of the point described in Example 1 at the end of 6 seconds.

SOLUTION. From (5) we have

$$\frac{d^2s}{dt^2} = \frac{9}{4}t.$$

Hence, by (6) the acceleration j at any time t is given by the equation

$$j = \tfrac{9}{4}t.$$

In particular, the desired acceleration at the end of 6 seconds will therefore be

$$j_1 = \tfrac{9}{4} \cdot 6 = 13\tfrac{1}{2} \text{ feet per second per second.}$$

55. Circular motion. The simplest type of motion, aside from the rectilinear type described in Art. 54, is circular motion. Here again the motion may be either uniform or non-uniform. If uniform, the central angle θ changes at a constant rate during the motion, this rate being known as the **angular velocity** of the moving point. If non-uniform, the angular velocity changes from point to point along the circle, its value at any one point being the value of the derivative $\frac{d\theta}{dt}$ at that point (compare Art. 54). Thus, we have the following law pertaining to all circular motion:

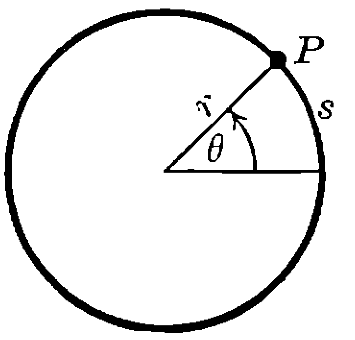

FIG. 67

$$\omega^* = \textit{angular velocity at } P = \frac{d\theta}{dt}, \qquad (7)$$

it being understood that the derivative $\frac{d\theta}{dt}$ is to be evaluated at the point P in question.

* The Greek letter ω is pronounced o-mā′ga.

The rate of change of the angular velocity in the case of non-uniform circular motion is called the **angular acceleration.** Thus, we have

$$\alpha = \textit{angular acceleration at } P = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \tag{8}$$

it being understood that the derivative is to be evaluated at the point P in question.

It is to be observed finally that if s represents the distance along the circular path from the starting point, which for simplicity we may take as corresponding to $\theta = 0$, then by elementary geometry we have $s = r\theta$ provided, however, that θ is measured in radians. Hence, at any time t we have

$$\frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega.$$

But, by reasoning as in Art. 54, it appears at once that $\frac{ds}{dt}$ measures the velocity of the moving point along the circle. We therefore have

$$v = \textit{velocity at } P = r\omega = r\frac{d\theta}{dt}, \tag{9}$$

it being understood that the radian is taken as the unit of angular measurement and that ω is to be evaluated at the point P in question.

Example. A point moves about on a circle of radius 6 inches, the law of its motion being

$$\theta = t^3 - 6\,t^2 + 9\,t, \tag{10}$$

where t is measured in seconds and θ in radians. Find (*a*) the angular velocity when $t = 4$ seconds; (*b*) the angular acceleration when $t = 1$ second; (*c*) the velocity of the point along the circle when $t = 5$ seconds; (*d*) the time and the angle at which the point reverses its direction of rotation (two answers); (*e*) the time and the angle at which the velocity changes from an increasing to a decreasing value, or vice versa.

SOLUTION. (*a*) From (10) we obtain

$$\omega = \frac{d\theta}{dt} = 3\,t^2 - 12\,t + 9. \tag{11}$$

Hence, by (7), the angular velocity when $t = 4$ seconds is

$$\omega_1 = 3\ \ 4^2 - 12 \cdot 4 + 9 = 9 \text{ radians per second.}$$

(*b*) From (11) we obtain

$$\alpha = \frac{d^2\theta}{dt^2} = 6\,t - 12. \tag{12}$$

Hence, by (8), the angular acceleration when $t = 1$ second is

$$\alpha_1 = 6 \cdot 1 - 12 = -6 \text{ radians per second per second,}$$

the significance of the negative sign being that the angular velocity is *decreasing* at the time $t = 1$ second.

(*c*) From (9) and (10) the velocity of the point at any time t is given by the formula

$$v = 6(3\,t^2 - 12\,t + 9).$$

Hence, the velocity when $t = 5$ seconds is

$$v_1 = 6(3\ 5^2 - 12 \cdot 5 + 9) = 144 \text{ feet per second.}$$

(*d*) The time at which the point reverses its direction of rotation is the time at which $\omega = 0$; that is, the value (or values) of t which satisfy the equation

$$3\,t^2 - 12\,t + 9 = 0.$$

Hence, solving, there are two such times; namely, $t_1 = 1$ second and $t_2 = 3$ seconds. Moreover, corresponding to these two values of t we obtain from (10) the values $\theta_1 = 4$ and $\theta_2 = 0$, so that the desired angles are $\theta_1 = 4$ radians and $\theta_2 = 0$.

(*e*) The time at which the velocity changes from an increasing to a decreasing value (or vice versa) is the time at which $\alpha = 0$, and this value of t is seen from (12) to be $t_1 = 2$ seconds, corresponding to which $\theta_1 = 2$ radians, as determined from (10).

EXERCISES

1. If a point moves along a straight line subject to the law

$$s = t^3 - 9\,t^2 + 24\,t + 1,$$

where the distance s is measured in feet and the time t is measured in seconds, answer the following questions, giving the proper interpretations to the signs of all results: (*a*) What is the velocity when $t = 3$ sec.*? (*b*) What is the velocity when $t = 5$ sec.? (*c*) What is the acceleration when $t = 1$ sec.? (*d*) What is the acceleration when $t = 5$ sec.? (*c*) At what times does the point change its direction of motion? (*f*) At what times does the point change its velocity from increasing to decreasing values, or vice versa?

2. If a point moves along a circle subject to the law

$$\theta = t^3 - 12\,t^2 + 45\,t + 2,$$

where the angle θ is measured in radians and the time t is measured in seconds, answer the following questions, giving the proper interpretations to the signs of all results: (*a*) What is the angular velocity when $t = 4$ sec.? (*b*) What is the angular velocity when $t = 6$ sec.? (*c*) What is the angular acceleration when $t = 2$ sec.? (*d*) What is the angular acceleration when $t = 6$ sec.? (*c*) At what time does the point change its direction of motion? (*f*) At what time does the point change its velocity from increasing to decreasing values, or vice versa?

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* Hereafter the abbreviations ft. and sec. will be used respectively for foot and second and the expression "feet per second per second" will be abbreviated into ft. per (sec.)^2.

3. By experiment it can be shown that a body when falling freely from rest near the earth's surface (the resistance of the air being neglected) obeys the law

$$s = \tfrac{1}{2} gt^2, \tag{1}$$

where g is a constant whose value, when the distance s is measured in feet and the time t is measured in seconds, is approximately 32.2. Hence establish the following laws relative to the velocity v and the acceleration j of such motion:

$$v = gt, \qquad \tfrac{1}{2} v^2 = gs, \tag{2}$$

$$j = g. \tag{3}$$

These laws are commonly known as the "laws of a falling body."

4. Assuming that a body falls from rest, find its velocity (a) at the end of 3 seconds; (b) at the end of 1 minute; (c) after it has fallen 10 feet.

5. If a body starts with any given velocity v_0 and moves along a straight line with any given constant acceleration j (positive or negative), it may be shown that the distance s traveled during the time t elapsed since the beginning of the motion is determined by the formula

$$s = v_0 t + \tfrac{1}{2} jt^2. \tag{1}$$

Hence show the existence of the following laws as regards the velocity v in all rectilinear motion in which the acceleration j is a constant:

$$v = v_0 + jt, \qquad \tfrac{1}{2} v^2 - \tfrac{1}{2} v_0^2 = js. \tag{2}$$

It is to be observed that in case $v_0 = 0$ and $j = g$ the laws (1) and (2) reduce, as they should, to (1) and (2) of Ex. 3.

6. An aeroplane, starting from rest, acquires a velocity of 100 miles per hour in 5 minutes. What is the acceleration (regarded as a constant)? See Ex. 5.

7. A train running 45 miles per hour is brought to rest uniformly in 3 minutes. (a) How great is the constant retardation (negative acceleration)? (b) How far does it travel? *Ans.* (a) 1320 ft. per (min.)^2; (b) 5940 ft.

8. A stone was dropped from a balloon while ascending at the rate of 30 feet per sec. and reached the ground in 10 sec. How high was the balloon when the stone was dropped? Take $g = 32.2$. *Ans.* 1310 ft.

9. A flywheel 8 ft. in diameter is making 40 revolutions per minute. What is its angular velocity and what is the velocity of a point upon its rim?

10. If a body starts moving in a circular path with a given initial angular velocity ω_0; that is, if $\omega = \omega_0$ when $t = 0$ and $\theta = 0$, and if the motion thereafter is constantly accelerated by an amount α (positive or negative), then it may be shown experimentally that the following relation between θ and t will hold true throughout the motion (compare (1), Ex. 5)

$$\theta = \omega_0 t + \tfrac{1}{2} \alpha t^2. \tag{1}$$

Hence show that the following laws exist as regards the angular velocity ω in all uniformly accelerated circular motion (compare (2), Ex. 5)

$$\omega = \omega_0 + \alpha t, \qquad \tfrac{1}{2} \omega^2 - \tfrac{1}{2} \omega_0^2 = \alpha\theta. \tag{2}$$

11. If the flywheel mentioned in Ex. 9 is brought uniformly to rest in $\frac{1}{3}$ of a minute by means of a brake, find (a) how great the constant retardation is and (b) how many revolutions the flywheel makes before it comes to rest. See Ex. 10.

Ans. (a) 240π rad. per (min.)^2; (b) $6\frac{2}{3}$ rev.

12. A wheel is revolving at the rate of 30 turns a second when a resistance begins to retard its motion uniformly at the rate of 6 radians per second. Find (*a*) in what time it will be brought to rest and (*b*) how many turns it will make in stopping. *Ans.* (*a*) 10π sec. (*b*) 150π rev.

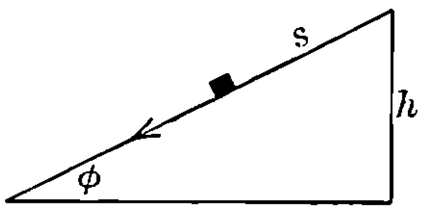

FIG. 68

13. If a body starting from rest slides down an inclined plane, friction being neglected, its motion obeys the law

$$s = \tfrac{1}{2} gt^2 \sin \phi,$$

where ϕ is the angle of inclination of the plane and where g is a constant which is equal approximately to 32.2, provided that the distance s is measured in feet and the time t is measured in seconds (compare Ex. 3). Hence establish the laws relative to the velocity in such motion and from your result show that the final velocity is the same as it would be if the body were to fall vertically through the height h of the plane.

14. Prove that the times of sliding from rest down the chords issuing from the highest point of a vertical circle are equal. See Ex. 13.

15. If the swing (amplitude) of an ordinary pendulum is small, it is shown in dynamics that the motion obeys approximately the law

$$\theta = \theta_0 \cos \sqrt{\frac{l}{g}}\, t,$$

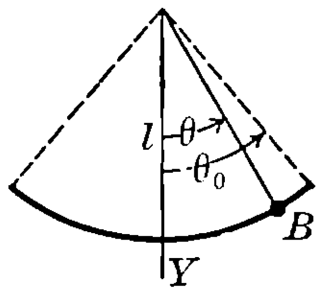

FIG. 69

where θ_0 is the angle (measured in radians) made by the rod OB with the vertical direction OY when the rod is in its outermost position, where l is the length of the rod, and where g is a constant whose value is approximately 32.2 in case l is measured in feet and t in seconds. Hence show that the velocity of the bob B is determined by the formula

$$v = l \sqrt{\frac{l}{g} (\theta_0^2 - \theta^2)},$$

and use this result to find the velocity of the bob when in its lowest position in case $\theta_0 = 10°$ and $l = 2$ ft.

56. General Curvilinear Motion. If a point moves along any given curve, its velocity v, when considered at any special position of the point such as P, is in the direction of the tangent to the curve at this point, while the magnitude of v at P may, by elementary physics, be regarded as the length of the diagonal of the rectangle whose dimensions are v_x and v_y, where v_x and v_y measure the magnitudes of the so-called "components" of the motion in the directions of the X- and Y-axes respectively at P. Hence, by elementary geometry,

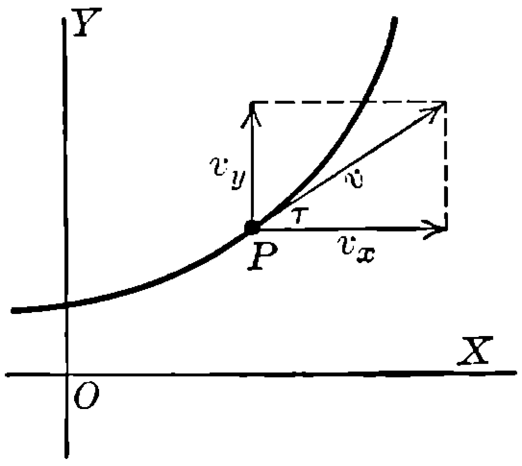

FIG. 70

$$v = \sqrt{v_x^2 + v_y^2}.$$

But v_x, being the rate of change of the abscissa x, is equal by Art. 54 to $\frac{dx}{dt}$, while for similar reasons $v_y = \frac{dy}{dt}$. Therefore, we reach the following universal formula for the magnitude of the velocity v of a moving point in any given position P upon the curve of motion:

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \tag{1}$$

it being understood that the derivatives are to be evaluated at the point P in question.

Note. The magnitude of v may also be regarded as the value of the expression $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, where Δs is the distance traversed along the curve beyond P in the time Δt. Relation (1) may, in fact, be shown to be a consequence of this latter universal definition of v.

Moreover, from Fig. 70 it is evident that, if τ represents the angle made by the direction of v at P with the positive X-axis, we have $\tan \tau = \frac{v_y}{v_x}$, or $\tau = \arc \tan \frac{v_y}{v_x}$. Hence, using the values of v_x and v_y as before, we obtain the formula

$$\tau = \textit{direction of velocity at } P = \text{arc tan} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \tag{2}$$

it being understood that the derivatives are to be evaluated at the point P in question.

Similarly, if we recall that the components parallel to the coördinate axes of the acceleration j are $j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$ and $j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$ and that, by elementary physics, j may be obtained from j_x and j_y by the same process as v may be obtained from v_x and v_y, we arrive at the formulas

$$j = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}, \tag{3}$$

$$\omega = \textit{direction of acceleration at } P = \textit{arc tan} \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}. \tag{4}$$

In order to obtain the derivatives which appear in the formulas (1)–(4) it is evidently necessary in any given case to know the abscissa x and the ordinate y of the moving point as functions of the time t. That is, the motion must be defined by two equations of the type

$$x = f(t), \qquad y = \phi(t). \tag{5}$$

It is also to be observed that by eliminating t from the two equations (5) we may obtain in any given case the equation of the path of the moving point in rectangular coördinates. Equations (5) are, in fact, the parametric equations of the path with t as parameter (Art. 51).

Example. A point (x, y) moves subject to the law

$$x = 2\,t^2, \qquad y = t^3, \tag{6}$$

x and y being measured in inches and t in seconds. For the point P reached at the end of 3 seconds find (*a*) the coördinates; (*b*) the magnitude of the velocity; (*c*) the direction of the velocity; (*d*) the magnitude of the acceleration; (*e*) the direction of the acceleration. Also obtain the equation of the path.

SOLUTION. (*a*) The position of the point when $t = 3$, as obtained by placing this value of t in (6), is seen to be (18, 27).

(*b*) From (6) we obtain

$$v_x = \frac{dx}{dt} = 4\,t, \qquad v_y = \frac{dy}{dt} = 3\,t^2. \tag{7}$$

Hence, by (1), the magnitude of the velocity when $t = 3$ is

$$v_1 = [\sqrt{16\,t^2 + 9\,t^4}]_{t=3} = \sqrt{873} = 29.55 \text{ in. per sec., approximately.}$$

(*c*) From (2) and (7) it appears that the direction of the velocity when $t = 3$ is

$$\tau_1 = \text{arc tan}\left[\frac{3\,t^2}{4\,t}\right]_{t=3} = \text{arc tan } 2\tfrac{1}{4} = \text{arc tan } 2.25 = 66°, \text{ approximately.}$$

(*d*) From (7) we obtain

$$\frac{d^2x}{dt^2} = 4, \qquad \frac{d^2y}{dt^2} = 6\,t. \tag{8}$$

Hence, by (3), the magnitude of the acceleration when $t = 3$ is

$$j_1 = [\sqrt{16 + 36\,t^2}]_{t=3} = \sqrt{340} = 18.44 \text{ in. per (sec.)}^2\text{, approximately.}$$

(*e*) From (4) and (8) it appears that the direction of the acceleration when $t = 3$ is

$$\omega_1 = \text{arc tan}\left[\frac{6\,t}{4}\right]_{t=3} = \text{arc tan } 4.5 = 77°\ 30', \text{ approximately.}$$

As regards the equation of the path, the second of the equations (6) gives $t = y^{\frac{1}{3}}$. Substituting this value of t in the first of the equations (6) and simplifying the result, the equation desired is seen to be

$$x^3 = 8\,y^2.$$

57. Tangential and Normal Acceleration. Formula (3) of Art. 56 gives what is termed in physics the *resultant* acceleration j of the moving point at any position P in the path. It is frequently desirable, however, to know also the components of j in the directions of the tangent and the normal at P. In this connection, we may now state and prove the following theorem:

Theorem. *The tangential and the normal components of the acceleration of a moving point at any given position P in its path are given respectively by the following formulas in which v represents the velocity and R represents the radius of curvature of the path*

$$j_t = \frac{dv}{dt}, \qquad j_n = \frac{v^2}{R}, \tag{1}$$

it being understood that these expressions are to be evaluated at the point P in question.

Proof. Representing by τ the angle made by the tangent at P with the positive X-axis, the acceleration in the tangential direction due to j_x alone will be $j_x \cos \tau$ while that due to j_y alone will be $j_y \sin \tau$. The sum of these two contributions will, therefore, be the total acceleration in the tangential direction. That is, we have

$$j_t = j_x \cos \tau + j_y \sin \tau.$$

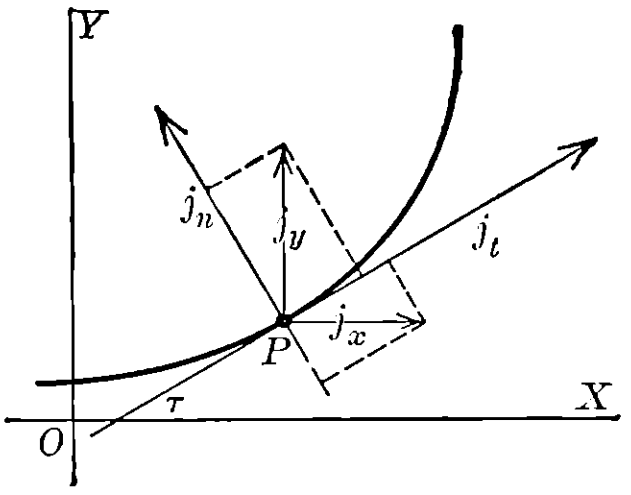

Fig. 71

But from Fig. 70 it appears that $\cos \tau = \frac{v_x}{v}$ and $\sin \tau = \frac{v_y}{v}$. Hence, it appears directly that the foregoing expression for j_t may be written in the form

$$j_t = \frac{v_x j_x + v_y j_y}{v}.$$

However, this expression may be identified as $\frac{dv}{dt}$, for we have

$$\frac{dv}{dt} = \frac{d}{dt}(v_x^2 + v_y^2)^{\frac{1}{2}} = \frac{2\, v_x \frac{dv_x}{dt} + 2\, v_y \frac{dv_y}{dt}}{2\sqrt{v_x^2 + v_y^2}} = \frac{v_x j_x + v_y j_y}{v}.$$

Thus the first of the two formulas (1) becomes established.

Again, we have by similar reasoning

$$\dot{j}_n = \dot{j}_y \cos \tau - \dot{j}_x \sin \tau = \frac{dv_y}{dt} \cdot \frac{v_x}{v} - \frac{dv_x}{dt} \cdot \frac{v_y}{v}. \tag{2}$$

But, with the meanings for s and R employed in Arts. 47–50 we may write

$$\frac{v_x}{v} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}} = \frac{\frac{dx}{d\tau} \cdot \frac{d\tau}{dt}}{\frac{ds}{d\tau} \cdot \frac{d\tau}{dt}} = \frac{\frac{dx}{d\tau}}{R}$$

and (3)

$$\frac{v_y}{v} = \frac{\frac{dy}{dt}}{\frac{ds}{dt}} = \frac{\frac{dy}{d\tau} \cdot \frac{d\tau}{dt}}{\frac{ds}{d\tau} \cdot \frac{d\tau}{dt}} = \frac{\frac{dy}{d\tau}}{R}.$$

Similarly we may write

$$\frac{dv_x}{dt} = \frac{dv_x}{d\tau} \cdot \frac{d\tau}{dt}, \qquad \frac{dv_y}{dt} = \frac{dv_y}{d\tau} \cdot \frac{d\tau}{dt}. \tag{4}$$

Substituting in (2) the expressions obtained in (3) and (4) for $\frac{v_x}{v}$, $\frac{v_y}{v}$, $\frac{dv_x}{dt}$, and $\frac{dv_y}{dt}$, we have

$$\dot{j}_n = \frac{1}{R}\left(\frac{dv_y}{d\tau} \cdot \frac{d\tau}{dt} \cdot \frac{dx}{d\tau} - \frac{dv_x}{d\tau} \cdot \frac{d\tau}{dt} \cdot \frac{dy}{d\tau}\right),$$

which in turn reduces to

$$\dot{j}_n = \frac{1}{R}\left(\frac{dv_y}{d\tau} \cdot \frac{dx}{dt} - \frac{dv_x}{d\tau} \cdot \frac{dy}{dt}\right) = \frac{1}{R}\left(v_x \frac{dv_y}{d\tau} - v_y \frac{dv_x}{d\tau}\right). \tag{5}$$

The expression appearing in the last parenthesis may now be identified as v^2, for we have by Art. 56

$$\tau = \text{arc} \tan \frac{v_y}{v_x},$$

so that, upon differentiating both members with respect to τ, we obtain by Art. 30

$$1 = \frac{1}{1 + \frac{v_y^2}{v_x^2}} \cdot \frac{v_x \frac{dv_y}{d\tau} - v_y \frac{dv_x}{d\tau}}{v_x^2} = \frac{v_x \frac{dv_y}{d\tau} - v_y \frac{dv_x}{d\tau}}{v^2}.$$

Hence

$$v_x \frac{dv_y}{d\tau} - v_y \frac{dv_x}{d\tau} = v^2,$$

so that (5) reduces to the indicated form, $j_n = \dfrac{v^2}{R}\cdot$

Thus, the second of the two formulas (1) becomes established and the proof becomes complete.

Example. If a point moves subject to the law described in the Example of Art. 56, find its tangential and its normal acceleration at the end of 3 seconds.

SOLUTION. From (7) of Art. 56 we have

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(4\,t)^2 + (3\,t^2)^2} = t\sqrt{16 + 9\,t^2}. \qquad (6)$$

Hence, by the foregoing theorem the tangential acceleration when $t = 3$ is

$$\left[\frac{d}{dt}\,t\sqrt{16 + 9\,t^2}\right]_{t=3} = \left[\frac{18\,t^2 + 16}{\sqrt{16 + 9\,t^2}}\right]_{t=3} = \frac{178}{\sqrt{97}}.$$
$$= 18.07 \text{ in. per (sec.)}^2\text{, approximately.}$$

In order to determine also the normal acceleration by use of the foregoing theorem, it is evidently necessary first to obtain the formula for the radius of curvature R of the path; that is, of the curve whose parametric equations (see (6), Art. 56) are

$$x = 2\,t^2, \qquad y = t^3.$$

This may be readily done by use of the theorems of Arts. 49–51, giving as the result

$$R = \frac{t}{12}[16 + 9\,t^2]^{\frac{3}{2}}. \qquad (7)$$

Using the expressions for v and R given in (6) and (7), the desired value of the normal acceleration when $t = 3$, as found by the foregoing theorem, is seen to be

$$\left[\frac{v^2}{R}\right]_{t=3} = \left[\frac{12\,t^2(16 + 9\,t)^2}{t(16 + 9\,t^2)^{\frac{3}{2}}}\right]_{t=3} = \left[\frac{12\,t}{\sqrt{16 + 9\,t^2}}\right]_{t=3} = \frac{36}{\sqrt{97}} = 3.65 \text{ in. per}$$
$$\text{(sec.)}^2\text{, approximately.}$$

EXERCISES

1. A point (x, y) moves in accordance with the law

$$x = t^2, \quad y = 2\,t,$$

where x and y are measured in feet and t in minutes. For the point P reached at the end of 2 minutes find (*a*) the coördinates; (*b*) the magnitude of the velocity; (*c*) the direction of the velocity; (*d*) the magnitude of the acceleration; (*e*) the direction of the acceleration; (*f*) the tangential acceleration; (*g*) the normal acceleration. Also obtain the equation in rectangular coördinates of the path and draw its graph.

Ans. (*a*) (4, 4); (*b*) $2\sqrt{5}$ ft. per min.; (*c*) arc tan $\frac{1}{2}$; (*d*) 2 ft. per (min.)^2; (*e*) 0°; $\frac{4}{5}\sqrt{5}$ ft. per (min.)^2; (*g*) $\frac{2}{5}\sqrt{5}$ ft. per (min.).^{2}

2. Work Ex. 1 in case the law of motion, instead of being as there stated, is

$$x = 2 \sin t, \quad y = 2 \cos t.$$

3. Neglecting the resistance of the air, it may be shown that the motion of a projectile obeys the law

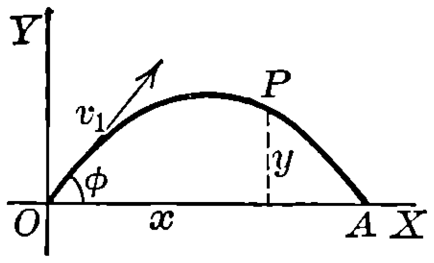

FIG.

$$x = v_1 \cos \phi \quad t, \quad y = v_1 \sin \phi \quad t - \tfrac{1}{2} gt^2,$$

where v_1 = the initial velocity, ϕ = the angle of projection with the horizon, t = the time of flight, and where g is a constant whose value, in case x and y are measured in feet and t in seconds, is 32.2 approximately. Find (*a*) the magnitude of the velocity at any time t; (*b*) the magnitude of the acceleration at any time t; (*c*) the magnitude and direction of the velocity when t = 2 seconds, it being given that v_1 = 100 ft. per sec. and $\phi = 30°$ Show also that the path of the projectile is a parabola.

4. Show that uniform circular motion of a point (x, y) is described by the equations

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \theta = ct,$$

where r and c are constants. Hence show that in uniform circular motion the acceleration is everywhere directed toward the center and is directly proportional to the radius.

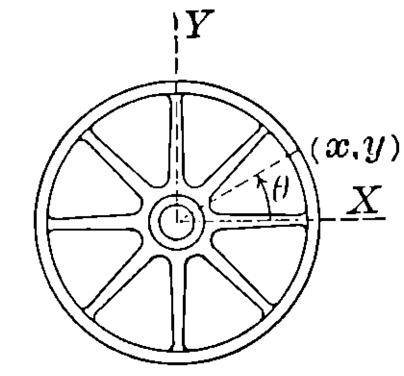

FIG. 73

5. It is a principle of dynamics that whenever a particle of mass m pounds moves along any given curve (as a heavy bead sliding along a wire or a planet revolving about the sun) the centrifugal force exerted by the particle at any given point P of the path is equal to $\frac{m}{g} j_n$ pounds, where j_n is the normal acceleration at P and where g is a constant whose value, in case the foot and the second are taken as the units of distance and time respectively, is 32.2 approximately. Hence show that if a particle of mass m revolves about an ellipse in accordance with the law

$$x = a \cos t, \quad y = b \sin t,$$

the centrifugal force possessed at either extremity of the major axis is the same at each revolution and is equal to $\frac{ma}{g}$, while the value of the centrifugal force at either extremity of the minor axis at each revolution is $\frac{mb}{g}$.

HINT. See Ex. 7, page 98.

6. If a point moves in accordance with the law

$$x = at, \quad y = b \sin at,$$

show that the acceleration at any instant is proportional to the distance of the point from the X-axis. Also describe the form and character of the path.

58. Miscellaneous Illustrations. Aside from the foregoing general considerations of this chapter it is to be noted that the calculus enables one to treat a wide class of miscellaneous problems dealing with rates. A few illustrative examples will best convey the nature of these problems and the methods of their solution.

Example 1. A man is walking at the rate of 6 miles per hour away from a tower 80 feet high standing in a horizontal field. At what rate is he receding from the top when 60 feet from the base?

SOLUTION. Let M be the man's position at the time t and suppose that at this time his distance from the base is x and his distance from the top is y. Then $y^2 = x^2 + (80)^2$, and since x and y are each functions of t, we obtain

$$2\,y\frac{dy}{dt} = 2\,x\frac{dx}{dt}, \quad \text{or} \quad \frac{dy}{dt} = \frac{x}{y}\cdot\frac{dx}{dt}. \tag{1}$$

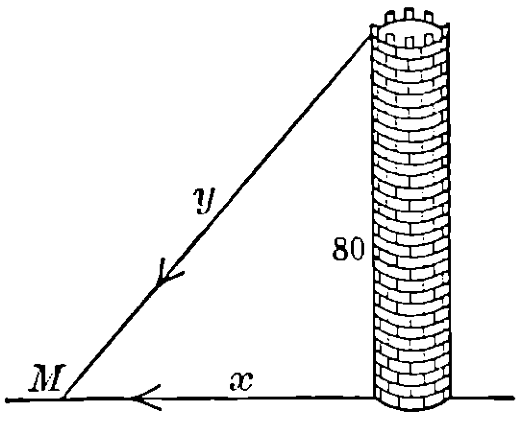

FIG. 74

Hence, at any time t the man is receding from the top x/y times as fast as he is receding from the base. But by hypothesis he is receding from the base at the rate of 6 miles per hour; that is, $dx/dt = 6 \times 5280$ ft. per hr. Thus (1) may be written

$$\frac{dy}{dt} = 6 \times 5280 \cdot \frac{x}{y}. \tag{2}$$

At the particular time with which we are concerned in the problem, namely, the time at which $x = 60$, we have $y = \sqrt{(60)^2 + (80)^2} = \sqrt{10{,}000} = 100$, so that (2) gives

$$\frac{dy}{dt} = 6 \times 5280 \times \frac{60}{100} \text{ ft. per hr.} = 3\tfrac{3}{5} \text{ mi. per hr.}$$

We conclude, therefore, that at the particular instant in question the man is receding from the top at the rate of $3\frac{3}{5}$ miles per hour.

Example 2. At a certain instant the base and the altitude of a triangle are respectively 8 inches and 10 inches in length and the former is increasing at the rate of 2 inches per second while the latter is decreasing at the rate of 3 inches per second. Is the area of the triangle increasing or decreasing at the instant in question and in either case by what amount?

SOLUTION. Let x be the length of the base, y the length of the altitude, and A the area at any time t. Then $A = \frac{1}{2}xy$ and hence, differentiating with respect to t,

$$\frac{dA}{dt} = \tfrac{1}{2}\,x\frac{dy}{dt} + \tfrac{1}{2}\,y\frac{dx}{dt}. \tag{3}$$

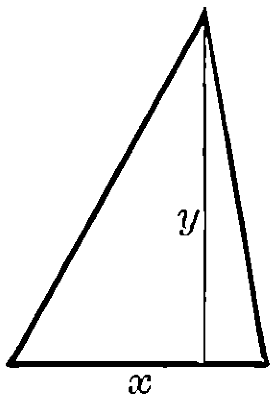

FIG. 75

But it is given that when $x = 8$ and $y = 10$, the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are respectively 2 and -3, the negative sign in the latter embodying the hypothesis that $\frac{dy}{dt}$ is decreasing. Thus, for $x = 8$, $y = 10$, we have

$$\frac{dA}{dt} = \tfrac{1}{2}\cdot 8 \cdot (-3) + \tfrac{1}{2}\cdot 10 \cdot 2 = -12 + 10 = -2 \text{ sq. in. per sec.}$$

Hence, the area of the triangle is *decreasing* at 2 square inches per second.

Example 3. Where in the first quadrant does the arc increase twice as fast as its sine?

SOLUTION. In Fig. 76 suppose that the radius of the circle is equal to 1 and that the arc s subtends the central angle θ, measured in radians. Then by trigonometry we shall have $y = \sin\theta$ and $\theta = s$. Hence, differentiating with respect to the time t, we shall have at any instant

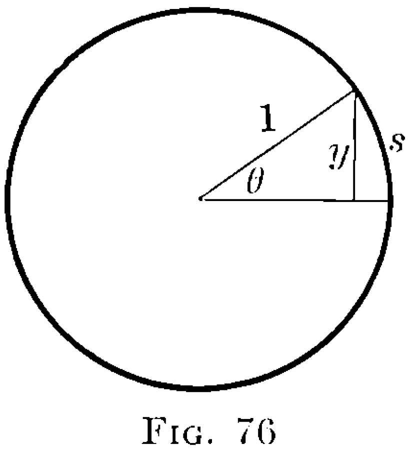

FIG. 76

$$\frac{dy}{dt} = \cos\theta\frac{d\theta}{dt} \text{ and } \frac{d\theta}{dt} = \frac{ds}{dt},$$

so that

$$\frac{dy}{dt} = \cos\theta \cdot \frac{ds}{dt}. \tag{4}$$

Here $\frac{dy}{dt}$ and $\frac{ds}{dt}$ represent respectively the rates at which the sine and its corresponding arc are increasing at any time t. At the particular time in question, namely, that at which $\frac{ds}{dt} = 2\frac{dy}{dt}$, equation (4) evidently gives $\cos\theta = \frac{1}{2}$, so that at this instant we must have by trigonometry $\theta = 60°$. This, therefore, is the angle desired.

EXERCISES

1. In Example 1 of Art. 58 find how fast the man is walking away from the tower when distant 50 feet from the base. *Ans.* 3.18 mi. per hr. approximately.

2. Work Example 2 of Art. 58 in case the base of the triangle is decreasing at the rate of 4 inches per second while the altitude is increasing at the rate of 3 inches per second, other conditions remaining the same.

Ans. Decreasing 8 sq. in. per sec.

3. In an expanding circle what will be the value of the radius when the area is increasing twice as fast as the radius? Answer the same question for the case in which the area and the radius are increasing at equal rates.

4. If at a certain time the two dimensions of a rectangle are a and b and these dimensions are diminishing at the rates m, n respectively, show that at the time in question the area of the rectangle is diminishing at the rate $an + bm$.

5. Having given that the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$ and that its surface is $4\pi r^2$, show that in an expanding sphere the volume at any instant is increasing $\frac{r}{2}$ times as fast as the area, r being the radius at the instant in question.

6. The three dimensions of a rectangular parallelepiped at a certain instant are 4 inches, 5 inches, and 6 inches, and these are increasing at the respective rates of 1 inch per second, 2 inches per second, and 3 inches per second. How fast is the volume increasing at the instant in question?

7. AB and CD are two straight railroad tracks, AB running due east and CD due north. At a certain time an eastbound train is traveling at a point 10 miles to the east of the crossroads, O, at the rate of 40 miles per hour, while a northbound train is traveling at a point 8 miles north of O at the rate of 50 miles per hour. How fast are the trains separating from each other at the instant in question?

Ans. 62.47 mi. per hr. approximately.

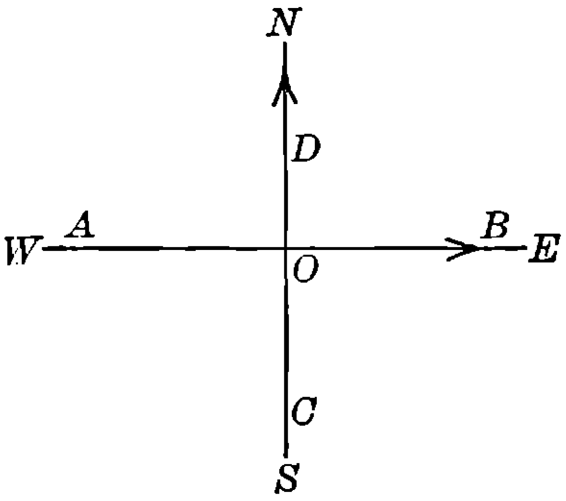

FIG.

8. Work Ex. 7 in case the northbound train is 8 miles south of O at the instant in question instead of 8 miles north of it, other conditions remaining the same.

9. A man 6 feet tall is walking away from an arc light suspended 20 feet from the ground. If the man's rate is 4 miles per hour, show (*a*) that the farther end of his shadow is receding along the ground at the constant rate of $5\frac{5}{7}$ miles per hour and (*b*) that the entire length of his shadow is increasing at the constant rate of $1\frac{5}{7}$ miles per hour.

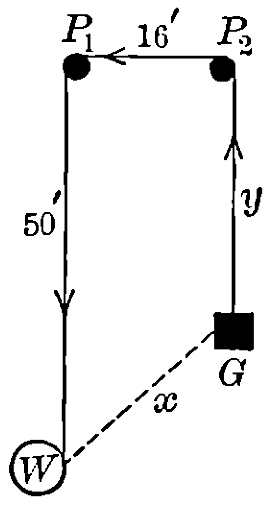

FIG. 78

10. In the accompanying figure W is a windlass which is hoisting a block of granite G by means of a rope passing over two pulleys P_1 and P_2, P_1 being 50 feet vertically above W while P_2 is 16 feet horizontally from P_1. If the rope is being pulled in at the rate of 5 feet per second, how fast is the top of the block receding from the windlass when at a distance of 38 feet from P_2?

Ans. 3 ft. per sec.

HINT. Neglect the size of the pulleys and of the drum of the windlass, thus regarding W, P_1, and P_2 as points.

11. A pail of mortar is being hoisted vertically along the side of a building by means of a rope passing over a pulley near the cornice. If the pulley is 80 feet above ground and the rope is being pulled in at the rate of 2 feet per second by a man who stands directly across the street at a distance of 40 feet from the building, how fast is the pail receding from the man when 40 feet above ground?

Ans. $\sqrt{2} = 1.414$ ft. per sec.

HINT. Neglect the size of the pulley and of the pail and disregard the distance of the man's hands from the ground; also neglect the small distance between the vertical rope and the building.

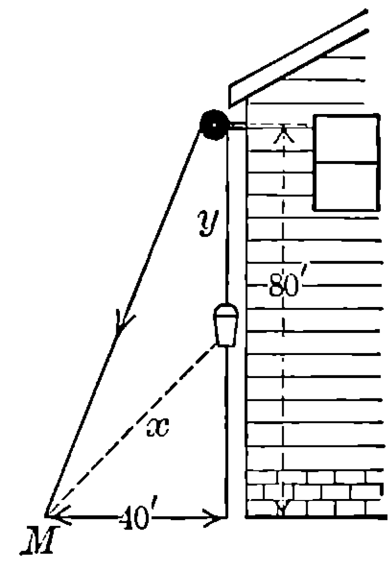

FIG. 79

12. One end of a ladder 30 feet long is leaning against the perpendicular side of a house. If the foot of the ladder be pulled away from the house at the rate of 3 feet per minute, find (*a*) how fast the top of the ladder will be descending when the foot is 18 feet from the house; (*b*) what will be the position of the foot when the top and bottom are moving at the same rate; (*c*) what will be the position of the top when descending at the rate of 6 feet per minute.

Ans. (*a*) $\frac{9}{4}$ ft. per min.; (*b*) $15\sqrt{2}$ ft.; (*c*) $6\sqrt{5}$ ft. from the ground.

13. The knob K on a door is 28 inches from the vertical axis AB through the hinges. If the door is opened at the rate of 10° per second, find (a) how fast the knob is receding from the doorway AD in the wall when the opening is 30°; (b) how fast the knob is approaching the rear wall AC when the opening is 75°.

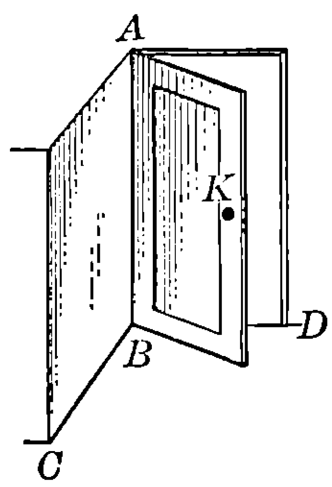

FIG. 80

Ans. (a) $\frac{7}{9}\sqrt{3}\,\pi$ in. per sec.; (b) $\frac{7\sqrt{2}}{18}(1+\sqrt{3})\pi$ in. per sec.

HINT. $Sin\,75° = \frac{\sqrt{2}}{4}(1+\sqrt{3})$.

14. A lighthouse containing a revolving reflector which makes 1 complete turn a minute is situated 4 miles off a straight shore. With what speed does the flash pass through a point on the shore 2 miles distant from the point on shore nearest the lighthouse? *Ans.* $10\,\pi = 31.416$ mi. per min.

15. A small boy is riding on a merry-go-round 40 feet in diameter and making 4 revolutions a minute while his father stands watching him at a distance of 10 feet from the revolving platform. How fast is the boy moving away from his father when one fourth the way round from him?

Ans. $\frac{480}{13}\sqrt{13}\,\pi = 418.23$ ft. per min. approximately.

16. An automobile wheel 32 inches in diameter is making 330 revolutions per minute as the car advances along a straight level road. Show that a point on the tire at a radius' distance from the center is advancing horizontally at the rate of $5(2+\sqrt{2})\pi$ miles per hour when its position angle, as indicated by ϕ in the adjoining figure, is 45° but that it is advancing at the (smaller) rate of $5(2-\sqrt{2})\pi$ miles per hour when the position angle is 135°. Does it make any difference whether the point in question lies on the forward half or the rear half of the wheel?

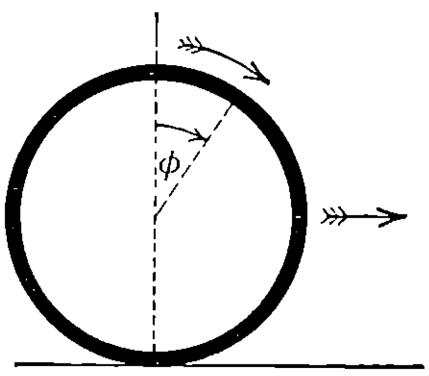

FIG. 81

HINT. Use the equations of the cycloid as given in Art. 52, noting that the angle θ there used and the angle ϕ just mentioned are connected by the relation $\theta = 180° + \phi$.

17. Prove that the topmost point of a rolling wheel travels forward just twice as fast as the center.

18. It is shown in solid geometry that if a small source of light is placed at the distance s from the surface of a sphere of radius r, the amount of the surface illuminated is given by the formula

$$I = \frac{2\pi r^2 s}{r+s}.$$

Hence show that if a man carrying a lantern is approaching a hemispherical haystack of diameter 30 feet at the rate of 5 feet per second, the amount of hay illuminated when he is 30 feet from the stack will be decreasing at the rate of about $26\frac{1}{4}$ square feet per second.

19. By means of the formula in Ex. 18 obtain a formula for the rate of increase of the visible area of a soap bubble as the bubble is blown up. Assume that the rate of increase of the radius is a constant, v, and that the distance of the eye from the bubble remains a constant, c.

20. Water is flowing from a faucet at the rate of 8 gallons a minute into a hemispherical bowl of radius 10 inches. How fast is the water level rising when the bowl is half full; that is, when the depth of the water is 5 inches?

Ans. 7.84 in. per min.

HINT. The volume of the smaller segment of a sphere of radius r cut off by a plane distant h from the center is given by the formula

$$V = \frac{\pi}{3}(r - h)^2(2r + h).$$

21. An elevated car is h feet vertically above a surface car, their tracks having directions which are at right angles to each other. If the speed of the elevated car is v_1 miles per hour while that of the surface car is v_2 miles per hour, show that the rate at which the two cars are separating t minutes after they meet is given by the formula

$$R = \frac{88\,t(v_1^2 + v_2^2)}{\sqrt{(88\,v_1t)^2 + (88\,v_2t)^2 + h^2}} \text{ mi. per hr.}$$

22. A projectile is moving in a parabola in accordance with the law stated in exercise 3, page 116. Show (*a*) that the velocity at any point in the path is that which would be acquired in falling freely to that point from the directrix of the parabola; (*b*) that, with a given initial velocity v_1 and a given accessible target A on the X-axis, there are two possible values for the angle ϕ and that their sum is 90°.

23. The center of a wagon wheel 2 feet in radius is traveling along a road with a velocity of 12 feet per second. If a piece of mud is thrown off the wheel at the top prove that it will never meet the circumference again.

24. A gun which can fire a projectile with a fixed initial velocity is on a plane inclined to the horizontal. Prove that the portion of the plane within range of the gun is bounded by an ellipse.

25. A lighted candle is carried directly away from the eye at a constant rate of speed. Show that when the candle is at the distance of 1 foot from the eye its brilliancy is diminishing 8 times as fast as when at the distance of 2 feet from the eye.

HINT. The intensity of a small source of light varies inversely as the square of the distance of the observer from it.

26. A pound weight is carried vertically away from the earth's surface at the constant rate of 1 foot per second. Show (*a*) that the rate at which it is losing its weight varies inversely as the cube of its distance from the earth's center and (*b*) that in the immediate neighborhood of the earth's surface its weight is diminishing at the rate of about 0.000000947 pounds per second.

HINT. By Newton's law of gravitation, the attraction of the earth varies inversely as the square of the distance from the earth's center.

27. If a point P moves around an ellipse whose center is at O in such a way that its eccentric angle increases at a constant rate c (see Fig. 64), show that its acceleration at any instant is directed along the line OP and that the amount of this acceleration is $c^2 \cdot \overline{OP}$.

CHAPTER VII

PARTIAL DERIVATIVES

59. Definition. Suppose that z is a function of the two independent variables x and y; that is, using the usual notation, suppose that

$$z = f(x, y).$$

Then the derivative of z with respect to x as x varies while y remains constant is called the **partial derivative of z with respect to x** and is denoted by $\frac{\partial z}{\partial x}$. Similarly, the derivative of z with respect to y as y varies while x remains constant is called **the partial derivative of z with respect to y** and is denoted by $\frac{\partial z}{\partial y}$.

Thus, if

$$z = x^2 + xy + y^2,$$

we have

$$\frac{\partial z}{\partial x} = 2\,x + y, \text{ } y \textit{ being here treated throughout as a constant,}$$

$$\frac{\partial z}{\partial y} = x + 2\,y, \text{ } x \textit{ being here treated throughout as a constant.}$$

In place of the symbols $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ other symbols are frequently used such as

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y};\ f_x'(x, y), f_y'(x, y);\ D_x f,\ D_y f.$$

The definitions just given may evidently be extended at once to functions of more than two independent variables. For example, if

$$u = f(x, y, z),$$

we have the three partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z};\ \text{or } \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z};\ \text{or } D_x f,\ D_y f,\ D_z f.$$

Thus, if

$$u = \sin(ax + by + cz),$$

we have

$\frac{\partial u}{\partial x} = a \cos(ax + by + cz)$, *both y and z being here regarded as constants;*

$\frac{\partial u}{\partial y} = b \cos(ax + by + cz)$, *both x and z being here regarded as constants;*

$\frac{\partial u}{\partial z} = c \cos(ax + by + cz)$, *both x and y being here regarded as constants.*

In the preceding illustration the function u is an explicit function of x, y, and z. As an illustration of the determination of partial derivatives of an implicit function, suppose for example that

$$u^2 = xy + yz + xz.$$

Then we have

$$2u\frac{\partial u}{\partial x} = y + z,\ 2u\frac{\partial u}{\partial y} = x + z,\ 2u\frac{\partial u}{\partial z} = y + x.$$

Hence we may write

$$\frac{\partial u}{\partial x} = \frac{y+z}{2u},\ \frac{\partial u}{\partial y} = \frac{x+z}{2u},\ \frac{\partial u}{\partial z} = \frac{y+z}{2u}.$$

60. Partial derivatives considered geometrically. When considered geometrically, every equation of the type $z = f(x, y)$, or more generally $f(x, y, z) = 0$, is the equation of a surface in space. Moreover, to keep y constant, say equal to b, implies that we are confining our attention to those points which lie upon the curve AB formed by the intersection of the surface by the plane $y = b$. The partial derivative $\frac{\partial z}{\partial x}$, therefore by Art. 9, measures the slope of the curve AB from point to point upon it. In other words, $\frac{\partial z}{\partial x}$ is equal to the tangent of the angle τ formed between the tangent line PT at any point P upon AB and the direction of the positive X-axis.

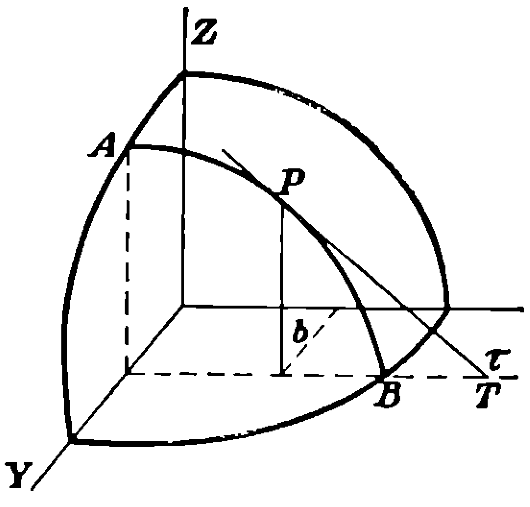

FIG. 82

Similarly, the partial derivative $\frac{\partial z}{\partial y}$ measures the slope of the curve formed by the intersection of the surface $f(x, y, z) = 0$ by a plane whose equation is of the type form $x = a$, where a is a constant, the slope-angle in this case being that between the tangent line to the curve and the direction of the positive Y-axis.

Example. Find the slope at the point (1, 2, 2) on the circle cut from the sphere

$$x^2 + y^2 + z^2 = 9 \tag{1}$$

by (*a*) the plane $y = 2$; (*b*) the plane $x = 1$.

SOLUTION. (*a*) Upon differentiating each member of (1) partially with respect to x, we obtain

$$2x + 0 + 2z\frac{\partial z}{\partial x} = 0.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{x}{z}.$$

At the given point (1, 2, 2), we thus have

$$\frac{\partial z}{\partial x} = -\frac{1}{2},$$

which is the slope required.

(*b*) Upon differentiating (1) partially with respect to y, we find readily that

$$\frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Hence at the given point (1, 2, 2) the required value of $\frac{\partial z}{\partial y}$ is -1.

The student is advised to draw a figure to illustrate these results.

61. Higher partial derivatives. The derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are themselves functions in general of x and y, hence they also have partial derivatives. These are denoted by the symbols $\frac{\partial^2 z}{\partial x^2}$ etc. as indicated in the following list

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

The meaning of still higher derivatives, such as $\frac{\partial^3 z}{\partial x^3}$, may be readily supplied by analogy with the foregoing.

It may be shown that, in general, we have the following relation whatever the function z:

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

That is, the order of differentiation in a partial derivative is immaterial. This fact will be assumed throughout what follows.

EXERCISES

1. If $z = x + y$, show that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 1.$

2. If $z = xy$, show that $\frac{\partial z}{\partial x} = y, \frac{\partial z}{\partial y} = x.$

3. If $z = x^2y + xy^2$, show that $\frac{\partial z}{\partial x} = 2\,xy + y^2, \frac{\partial z}{\partial y} = x^2 + 2\,xy.$

4. If $z = \frac{xy}{x + y}$, show that $\frac{\partial z}{\partial x} = \frac{y^2}{(x + y)^2}, \frac{\partial z}{\partial y} = \frac{x^2}{(x + y)^2}.$

5. If $z = \sin(x + y)$, show that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \cos(x + y).$

6. If $z = \tan\frac{x}{y}$, show that $\frac{\partial z}{\partial x} = \frac{1}{y}\sec^2\frac{x}{y}, \frac{\partial z}{\partial y} = -\frac{x}{y^2}\sec^2\frac{x}{y}.$

7. If $w = \text{arc tan}\,(p + q)$, show that $\frac{\partial w}{\partial p} = \frac{\partial w}{\partial q} = \frac{1}{1 + (p + q)^2}.$

8. If $s = r^t$, show that $\frac{\partial s}{\partial r} = tr^{t-1}, \frac{\partial s}{\partial t} = r^t \log r.$

9. If $z = \log(x^2 + y^2)$, show that $\frac{\partial z}{\partial x} = \frac{2\,x}{x^2 + y^2}, \frac{\partial z}{\partial y} = \frac{2\,y}{x^2 + y^2}.$

10. If $u = xy + xz + yz$, show that $\frac{\partial u}{\partial x} = y + z, \frac{\partial u}{\partial y} = x + z, \frac{\partial u}{\partial z} = x + y.$

11. If $u = \sin(x + y + z)$, show that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}.$

12. If $u = \log(x^2 + y^2 + z^2)$, show that $\frac{\partial u}{\partial x} \; \frac{\partial u}{\partial y} \; \frac{\partial u}{\partial z} = x : y$

13. If $x^2 - y^2 + z^2 = 25$, show that $\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = \frac{y}{z}.$

HINT. See Art. 34; also the solution of the Example in Art. 60.

14. If $x^2y^2 + x^2z^2 + y^2z^2 = a^2$, show that $\frac{\partial z}{\partial x} = -\frac{x(y^2 + z^2)}{z(x^2 + y^2)}, \frac{\partial z}{\partial y} = -\frac{y(x^2 + z^2)}{z(x^2 + y^2)}.$

15. If $x^3 + y^3 + z^3 = 3\,xyz$, show that $\frac{\partial z}{\partial x} = \frac{yz - x^2}{z^2 - xy}, \frac{\partial z}{\partial y} = \frac{xz - y^2}{z^2 - xy}.$

16. If $z = \sin xyz$, show that $\frac{\partial z}{\partial x} = \frac{yz \cos xyz}{1 - xy \cos xyz}, \frac{\partial z}{\partial y} = \frac{xz \cos xyz}{1 - xy \cos xyz}.$

17. If $z = \log(x + y)$, show that $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x + y)^2}.$

18. If the surface whose equation is $xy + xz + yz = 1$ is cut by the plane $y = 1$ and a tangent is drawn to the curve of intersection at the point (1, 1, 0) upon it, find, correct to 10 minutes, the angle at which the tangent will meet the XY-coördinate plane. *Ans.* 153° 30′.

19. Work Ex. 18 in case the intersecting plane used is $x = 1$ instead of $y = 1$, other conditions remaining the same.

20. Work Exs. 18 and 19 in case the surface used is $x^3 + y^3 + z^3 - 3\,xyz = 2$, other conditions remaining the same. *Ans.* 45° in each case.

21. If $z = x^3y^2 + x^2y^3$, find $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y\,\partial x}, \dfrac{\partial^2 z}{\partial x\,\partial y}, \dfrac{\partial^2 z}{\partial y^2}$.

22. Given $z = \sin xy$. Verify the relation $\dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{\partial^2 z}{\partial x\,\partial y}$.

23. If $z = e^{xy}$, verify the relation $\dfrac{\partial^3 z}{\partial y\,\partial x^2} = \dfrac{\partial^3 z}{\partial x\,\partial y\,\partial x} = \dfrac{\partial^3 z}{\partial x^2\,\partial y}$.

24. If $u = (y - z)(z - x)(x - y)$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

25. If $u = x^2 + y^2 + yz$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\,u$.

26. If $u = \log(x^2 + y^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

27. If $u = e^{xy}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{u}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2\right]$.

28. If $u = \sin(x + y + z)$, show that

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial y^3} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial^3 u}{\partial z^3} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z} = -\,u.$$

29. If $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

62. Tangent plane and normal line to a surface. We may now state and prove the following theorem (compare Art. 45):

Theorem I. *The equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_1, y_1, z_1) upon this surface is given by the formula*

$$z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1),$$

where $\dfrac{\partial z_1}{\partial x_1}$ and $\dfrac{\partial z_1}{\partial y_1}$ indicate the values of the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ respectively when $x = x_1$, $y = y_1$, and $z = z_1$.

PROOF. From solid analytic geometry we know that every plane passing through the point (x_1, y_1, z_1) has an equation of the type form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0, \tag{1}$$

where A, B, and C are constants. Now, it follows from Art. 60 that (1) will be tangent to the surface

$$z = f(x, y) \tag{2}$$

at the point (x_1, y_1, z_1) if, and only if, the partial derivatives $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ as determined from (1) have the same values at (x_1, y_1, z_1) as the deriva-

tives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ as determined by (2) when evaluated at the same point. But the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ as determined from (1) are readily found to be $-\frac{A}{C}$ and $-\frac{B}{C}$ respectively. It follows, then, that (1) will be the tangent plane to (2) at (x_1, y_1, z_1) if

$$\frac{\partial z_1}{\partial x_1} = -\frac{A}{C} \quad \text{and} \quad \frac{\partial z_1}{\partial y_1} = -\frac{B}{C}, \tag{3}$$

where $\frac{\partial z_1}{\partial x_1}$ and $\frac{\partial z_1}{\partial y_1}$ indicate the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (x_1, y_1, z_1) as determined from (2).

From (3) we have $A = -C\frac{\partial z_1}{\partial x_1}$, $B = -C\frac{\partial z_1}{\partial y_1}$, and, introducing these values for A and B in (1) and then dividing the resulting equation through by $-C$, we obtain as the equation of the tangent plane

$$\frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1) - (z - z_1) = 0, \tag{4}$$

which may be written in the form given in the theorem.

The proof is thus complete.

Note. The theorem fails in case the tangent plane in question is perpendicular to the XY-coördinate plane, for then we have $C = 0$ in (1), as follows from elementary solid analytic geometry, and with this value of C the steps of the foregoing proof cannot be carried through inasmuch as they involve division by zero.

We may now also state and prove the following theorem:

Theorem II. *The equation of the normal line to the surface $z = f(x, y)$ at the point (x_1, y_1, z_1) upon this surface is given by the formula*

$$\frac{x - x_1}{\frac{\partial z_1}{\partial x_1}} = \frac{y - y_1}{\frac{\partial z_1}{\partial y_1}} = \frac{z - z_1}{-1},$$

where $\frac{\partial z_1}{\partial x_1}$ and $\frac{\partial z_1}{\partial y_1}$ indicate the values of the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial x}{\partial y}$ respectively when $x = x_1$, $y = y_1$, and $z = z_1$.

PROOF. Upon observing that the coefficients of x, y, and z in the equation of the tangent plane (4) are respectively $\frac{\partial z_1}{\partial x_1}, \frac{\partial z_1}{\partial y_1}$, and -1, it follows immediately from solid analytic geometry that the equation of the line perpendicular to this plane at the point (x_1, y_1, z_1) is as stated in the theorem.

Example. Find the equation of the tangent plane and the normal line to the ellipsoid

$$\frac{x^2}{8} + \frac{y^2}{4} + \frac{z^2}{2} = 1 \tag{4}$$

at the point $(\sqrt{2}, 1, -1)$ upon its surface.

SOLUTION. By differentiating (4) member by member partially with respect to x and partially with respect to y, we find readily that

$$\frac{\partial z}{\partial x} = -\frac{x}{4z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{2z}.$$

At the given point $(\sqrt{2}, 1, -1)$ we therefore have

$$\frac{\partial z_1}{\partial x_1} = -\frac{\sqrt{2}}{-4} = \frac{\sqrt{2}}{4}, \quad \frac{\partial z_1}{\partial y_1} = -\frac{1}{-2} = \frac{1}{2}.$$

By use of the first theorem, the equation of the tangent plane at the given point thus becomes

$$z + 1 = \frac{\sqrt{2}}{4}(x - \sqrt{2}) + \frac{1}{2}(y - 1),$$

which may be reduced to the form

$$\sqrt{2}\,x + 2\,y - 4\,z - 8 = 0.$$

Again, by means of the second theorem the equation of the normal at the given point becomes

$$\frac{x - \sqrt{2}}{\frac{\sqrt{2}}{4}} = \frac{y - 1}{\frac{1}{2}} = \frac{z + 1}{-1}, \quad \text{or} \quad \frac{x - \sqrt{2}}{\sqrt{2}} = \frac{y - 1}{2} = \frac{z + 1}{-4}.$$

63. Total derivatives. It frequently happens that a function z is given as a function of two variables x and y and in addition to this the variables x and y are themselves functions of some single variable, as t. In this case z is clearly a function of t alone, and we shall now show that its derivative with respect to t may be calculated by means of the following theorem which may be regarded as a generalization of the theorem of Art. 16. In all such cases the derivative in question, $\frac{dz}{dt}$, is called the **total derivative** of z, thus distinguishing it from the partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$:

Theorem. *If $z = f(x, y)$ where x and y are given as functions of the variable t, then the total derivative $\frac{dz}{dt}$ may be calculated by means of the formula*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

PROOF. It is first to be observed that for any given value of t the corresponding values of both x and y and hence that of z are fully determined, as results from the present hypotheses. Stated geometrically, this implies that to each value of t there corresponds a determinate point $P = (x, y, z)$ upon the given surface $z = f(x, y)$. Now, supposing a value of t to have been chosen, let us suppose that t is given an increment Δt. Corresponding to the new value $t + \Delta t$ there will be a second point Q upon the surface. Suppose that the coördinates of this point are $Q = (x + \Delta x, y + \Delta y, z + \Delta z)$. Geometrically considered, Δx and Δy will then be certain line-segments in the XY plane as indicated in Fig. 83, while Δz will be the line segment LQ, it being understood that the plane PSL has been drawn parallel to the XY plane. From the definition of a derivative we then have

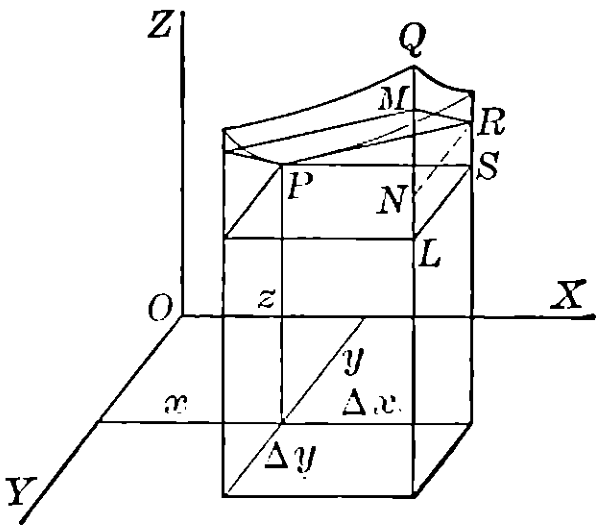

FIG. 83

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \frac{LQ}{\Delta t}, \qquad (1)$$

and it remains but to study the limit here appearing on the right.

Draw the tangent plane PRM to the surface at P. Then

$$\lim_{\Delta t \to 0} \frac{LQ}{\Delta t} = \lim_{\Delta t \to 0} \frac{LM}{\Delta t}.* \qquad (2)$$

But $LM = LN + NM = SR + NM$. Moreover, (3)

$SR = PS \tan \angle SPR = \Delta x \dfrac{\partial f}{\partial x}$, $NM = NR \tan \angle NRM = \Delta y \dfrac{\partial f}{\partial y}$. (Art. 70)

Thus (3) may be written

$$LM = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \qquad (4)$$

and hence we have

$$\frac{LM}{\Delta t} = \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta t}. \qquad (5)$$

Upon taking the limit as $\Delta t \to 0$ of each member of (5), recalling also relations (1) and (2), we obtain as desired

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

* This relation, which we here regard as self-evident, may be proven to be correct, at least under slightly limiting conditions upon the function $f(x, y)$. Such considerations, however, pertain properly to a second course in calculus.

Corollary. The theorem may be readily generalized so as to apply to a function of any number of variables. Thus, in the case of a function $u = f(x, y, z)$ of the three variables x, y, z, each of which is a given function of t, the formula becomes

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example 1. Given $z = e^{ax} \sin y$ in which $x = t^2$ and $y = e^t$. Find $\frac{dz}{dt}$.

SOLUTION. We have $\frac{\partial z}{\partial x} = ae^{ax} \sin y$, $\frac{\partial z}{\partial y} = e^{ax} \cos y$, $\frac{dx}{dt} = 2\,t$, $\frac{dy}{dt} = e^t$. The theorem therefore gives

$$\frac{dz}{dt} = ae^{ax} \sin y \quad 2\,t + e^{ax} \cos y \quad e^t = e^{at^2}(2\,at \sin e^t + e^t \cos e^t).$$

Example 2. The three dimensions of a rectangular parallelepiped are 5, 4, and 3 inches respectively. At a certain instant they are increasing at the rate of 2 inches, 1 inch, and 6 inches per second respectively. At what rate is the volume of the cube increasing at that instant?

SOLUTION. Let the three (variable) dimensions be x, y, z and let u be the volume. Then $u = xyz$. Hence, $\frac{\partial u}{\partial x} = yz$, $\frac{\partial u}{\partial y} = xz$, $\frac{\partial u}{\partial z} = xy$, so that by the foregoing corollary we may write

$$\frac{du}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt}.$$

But by hypothesis we have at the instant in question $x = 5$, $y = 4$, $z = 3$, and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 1$, $\frac{dz}{dt} = 6$. At this instant we therefore have

$$\frac{du}{dt} = 12 \cdot 2 + 15 \cdot 1 + 20 \cdot 6 = 159 \text{ cu. in. per sec.} \quad \textit{Ans.}$$

64. Differentiation of implicit functions. We may now state and prove the following important theorem concerning the derivative of any implicit function y of the variable x.

Theorem. *If y is defined implicitly as a function of x through the equation $f(x, y) = 0$, the derivative $\frac{dy}{dx}$ is determined by the formula*

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

PROOF. For convenience, place $z = f(x, y)$. Then, by the theorem of Art. 63 we may write

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}. \qquad (1)$$

But under the present hypotheses we have $z = 0$ for all values of x, and hence also $\frac{dz}{dx} = 0$. Equation (1) thus becomes

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}.$$

Upon solving this equation for $\frac{dy}{dx}$ we obtain directly the form for it given in the theorem, thus completing the proof.

Example. Find $\frac{dy}{dx}$ if $x^3y^2 + \sin y = 0$.

SOLUTION. Let $f(x, y) = x^3y^2 + \sin y$. Then, $\frac{\partial f}{\partial x} = 3x^2y^2$, $\frac{\partial f}{\partial y} = 2x^3y + \cos y$. Hence, by the theorem, we have

$$\frac{dy}{dx} = -\frac{3x^2y^2}{2x^3y + \cos y}.$$

This result, here obtained by a general method, may be obtained though with more labor, by the special methods illustrated in Art. 34, as the student may readily verify.

EXERCISES

1. Find the equation of the tangent plane and the equations of the normal line to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1, -2, -3)$.

Ans. $x - 2y - 3z = 14$; $6x = -3y = -2z$.

2. Find the equation of the tangent plane and the equations of the normal line to the surface (hyperboloid of one sheet) $\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{1} = 1$ at the point $(3, 2, -1)$.

Ans. $2x - 3y - 6z = 6$; $\frac{x-3}{2} = \frac{y-2}{-3} = \frac{z+1}{-6}$.

3. Find the equation of the tangent plane and the equations of the normal line to the surface (ellipsoid) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_1, y_1, z_1).

Ans. $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} + \frac{z_1z}{c^2} = 1$; $\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1} = \frac{c^2(z - z_1)}{z_1}$.

4. By use of the theorem and the corollary of Art. 63 establish the following statements:

(*a*) If $u = x^2y$ and $x = 1 + t$, $y = t^2$,

then $\dfrac{du}{dt} = 2\,xy + 2\,tx^2 = 2\,t(1 + t)(2\,t + 1).$

(*b*) If $u = xyz$ and $x = e^t$, $y = \log t$, $z = \sin t$,

then $\dfrac{du}{dt} = yze^t + \dfrac{xz}{t} + xy \cos t.$

5. By use of the theorem and corollary of Art. 63 find $\dfrac{du}{dt}$ in case

(*a*) $u = (1 - x)(1 + y)$, $x = t^2$, $y = t^3$;

(*b*) $u = \sqrt{(1 - x)(1 + y)}$, $x = t^2$, $y = t^3$;

(*c*) $u = \sin (xy^2)$, $x = 1 + t$, $y = t^2$;

(*d*) $u = \text{arc} \tan \dfrac{x}{y}$, $x = e^t$, $y = \sin t$;

(*e*) $u = \log (x^2 + y^2 + z^2)$, $x = t$, $y = t^2$, $z = t^3$;

(*f*) $u = \sin \sqrt{x^2 + y^2 + z^2}$, $x = \log t$, $y = \dfrac{1}{t}$, $z = 1 - t$.

6. Given that the area A of an ellipse is determined by the formula $A = \pi ab$, where a and b are respectively the semimajor and semiminor axes. If, at a certain instant, the (entire) major axis is 6 inches long and is increasing at the rate of 2 inches per second and the minor axis is 4 inches long and is decreasing at the rate of 2 inches per second, is the area of the ellipse increasing or decreasing at the instant in question, and at what rate? *Ans.* Decreasing at π sq. in. per sec.

7. The volume V of an ellipsoid is given by the formula $V = \frac{4}{3}\pi abc$, where a, b, and c are the three semiaxes. If, at a certain instant, the three axes are equal respectively to 12, 8, and 6 inches and of these the first is increasing at the rate of 4 inches per second, the second is increasing at the rate of 6 inches per second, and the third is decreasing at the rate of 4 inches per second, find the rate at which the volume is increasing (or decreasing) at the instant in question.

Ans. Increasing $40\,\pi$ cu. in. per sec.

8. The height of a right circular cone is 10 inches and is increasing at the rate of 3 inches per second. At the same instant the radius of the base is 4 inches and is decreasing at the rate of 2 inches per second. At the instant in question find (*a*) whether the volume is increasing or decreasing and at what rate; (*b*) whether the lateral area is increasing or decreasing and at what rate; (*c*) whether the total area is increasing or decreasing and at what rate.

Ans. (*a*) Decreasing $37\frac{1}{3}\,\pi$ cu. in. sec.; (*b*) decreasing $\frac{72}{29}\sqrt{29}\,\pi$ sq. in. sec.; (*c*) decreasing $(\frac{72}{29}\sqrt{29} + 16)\,\pi$ sq. in. sec.

9. By means of the theorem of Art. 64 find the derivative $\dfrac{dy}{dx}$ for each of the following implicit functions y:

(*a*) $x^2 - y^2 + xy^3 = 0$;

(*b*) $(2\,x - 3\,y^2)^3 = xy + 1$;

(*c*) $\sin \dfrac{x}{y} = x$;

(*d*) $e^{\bar{y}} = y$;

(*e*) $\log (x^2 + y^2) = x + y$;

(*f*) $\cos^2 (x^2 - y^2) = x^2 + y^2$.

10. By use of the theorem of Art. 64 show that if z is an implicit function of the two independent variables x, y defined by the equation $F(x, y, z) = 0$, the partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are determined by the formulas

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

11. Show that the tangent plane to the sphere

$$x^2 + y^2 + z^2 + 2\,Lx + 2\,My + 2\,Nz + D = 0$$

at the point (x_1, y_1, z_1) is given by the formula

$$x_1x + y_1y + z_1z + L(x + x_1) + M(y + y_1) + N(z + z_1) + D = 0.$$

12. Find the equation of the tangent plane at the point (x_1, y_1, z_1) to the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ and show that the sum of the squares of the intercepts of this plane on the coördinate axes is a constant.

13. Prove that the tetrahedron formed by the coördinate planes and any tangent plane to the surface $xyz = a^3$ is of constant volume.

14. The angle at which two surfaces intersect at a point P is defined to be the angle between the normals to the two surfaces at this point, this angle being equal to the angle of intersection of the tangent planes to the two surfaces at P. Hence, find correct to 10 minutes, the angle at which the plane $x + y + z = 2$ meets the sphere $x^2 + y^2 + z^2 = 6$ at the point $(2, 1, -1)$.

HINT. If l_1, l_2, l_3 be the direction cosines of one line and m_1, m_2, m_3 be the direction cosines of another line, the angle θ between the two lines is determined by the formula

$$\cos\theta = l_1m_1 + l_2m_2 + l_3m_3.$$

15. Determine a and b so that the hyperbolic paraboloid $8\,z = ax^2 + by^2$ may intersect the ellipsoid $x^2 + 2\,y^2 + 3\,z^2 = 12$ at right angles at the point $(1, 2, -1)$.

16. Show that the sphere $x^2 + y^2 + z^2 = 3$ is tangent to the surface $xyz = 1$ at the point $(1, 1, 1)$.

HINT. Two surfaces are tangent to each other at a point P if they have a common normal at that point.

17. If the two surfaces $F(x, y, z) = 0$ and $\Phi(x, y, z) = 0$ intersect at the point $P = (x_1, y_1, z_1)$, show (a) that the intersection will be at right angles if, and only if,

$$\frac{\partial F_1}{\partial x_1}\frac{\partial \Phi_1}{\partial x_1} + \frac{\partial F_1}{\partial y_1}\frac{\partial \Phi_1}{\partial y_1} + \frac{\partial F_1}{\partial z_1}\frac{\partial \Phi_1}{\partial z_1} = 0,$$

where $\frac{\partial F_1}{\partial x_1}$ denotes the value of $\frac{\partial F}{\partial x}$ when $x = x_1$, $y = y_1$, $z = z_1$, etc.; (b) that the two surfaces will be tangent to each other at P if, and only if,

$$\frac{\frac{\partial F_1}{\partial x_1}}{\frac{\partial \Phi_1}{\partial x_1}} = \frac{\frac{\partial F_1}{\partial y_1}}{\frac{\partial \Phi_1}{\partial y_1}} = \frac{\frac{\partial F_1}{\partial z_1}}{\frac{\partial \Phi_1}{\partial z_1}}.$$

CHAPTER VIII

SINGULAR POINTS. ASYMPTOTES. CURVE TRACING

65. Introduction. It has been shown in earlier chapters, especially Chap. V, that the derivative of a function has immediate applications to geometry. The results heretofore noted, however, form but a beginning to this subject. In the present chapter we shall note various further geometric applications, including applications of partial derivatives.

66. Definitions. Let AB be an arc of any curve. If at the point P upon AB it is possible to draw one and only one tangent, then P is said to be an **ordinary point** upon the curve. If, on the other hand, it is possible to draw at P two and only two distinct tangents, then P is called a **node.** In case the two tangents must be regarded as coincident, the point P in question is called a **cusp,** several possibilities being then present. Thus, if the curve in the neighborhood of P lies on both sides of the common tangent, as in Fig. 86, P is called a cusp of the **first kind;** if the curve lies on but one side of the common tangent, as in Fig. 87, P is called a cusp of the **second kind;** if the curve has two distinct cusps at P, as in Fig. 88, P is called a **double cusp,** or a **point of osculation.** Moreover, any point P for which the equations of two and only two tangents, distinct or coincident, real or imaginary, exist in accordance with the formula of Art. 45 is called a **double point** of the curve. Nodes and cusps are, therefore, but special kinds of double points, being double points at which the two tangents are real and may therefore be actually drawn.

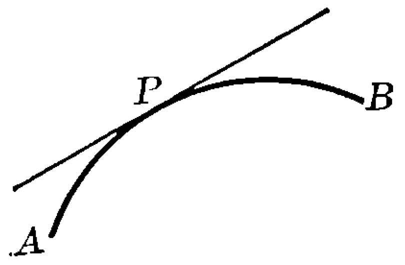

Fig. 84
Ordinary Point

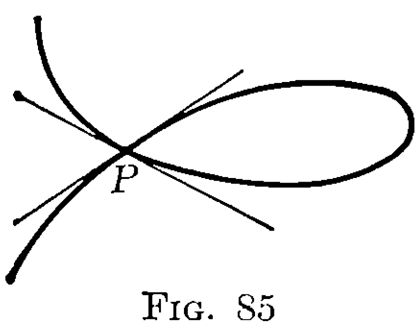

Fig. 85
Node

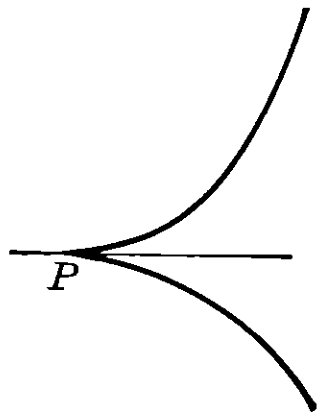

Fig. 86
Cusp of First Kind

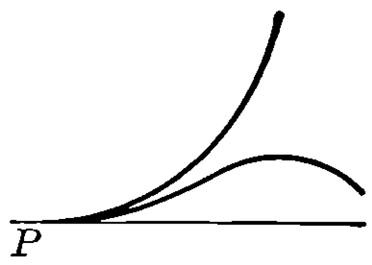

Fig. 87
Cusp of Second Kind

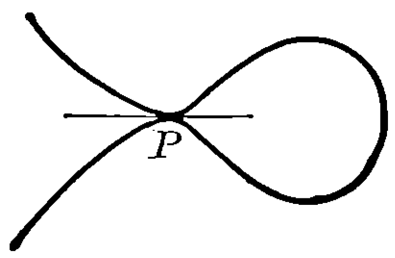

Fig. 88
Point of Osculation

A point for which the equations of three and only three tangents, distinct or otherwise, real or imaginary, exist in accordance with the formula of Art. 45 is called a **triple point.** Similarly, the definition of a **quadruple point, quintuple point,** etc., may be supplied.

All points of a curve other than ordinary points are known as **singular points.**

67. Location of singular points. We may now state and prove the following theorem:

Theorem. *The point* (x_1, y_1) *can be a singular point of the curve* $f(x, y) = 0$ *only in case the first partial derivatives* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *(assumed finite at* (x_1, y_1)*) are each equal to zero at this point; that is, only in case*

$$\frac{\partial f_1}{\partial x_1} = 0 \quad \textit{and} \quad \frac{\partial f_1}{\partial y_1} = 0.$$

PROOF. At (x_1, y_1) the value of the derivative $\frac{dy}{dx}$, as given by the theorem of Art. 64, is

$$\frac{dy_1}{dx_1} = -\frac{\frac{\partial f_1}{\partial x_1}}{\frac{\partial f_1}{\partial y_1}}. \tag{1}$$

This fraction will have definite significance, thus enabling the equation of a single tangent at (x_1, y_1) to be written by the theorem of Art. 45, except when $\frac{\partial f_1}{\partial x_1} = 0$ and $\frac{\partial f_1}{\partial y_1} = 0$. In this failing case the fraction becomes $-\frac{0}{0}$ and hence is indeterminate (see Art. 6), thus allowing no inference to be drawn as to the value of $\frac{dy_1}{dx_1}$ and hence no inference as to the number or character of the tangents at (x_1, y_1). It is, therefore, only when this failing case is present that the equation of more than one tangent can be possible; that is, it is only then that a singular point can be present.

Note. It may be observed that in case $\frac{\partial f_1}{\partial x_1}$ is not zero but $\frac{\partial f_1}{\partial y_1} = 0$ the value of $\frac{dy_1}{dx_1}$ as given by (1) becomes infinite. However, this merely signifies geometrically that the tangent at (x_1, y_1) is perpendicular to the X-axis, its equation then becoming simply $x = x_1$.

Example. Locate all possible singular points upon the curve (semi-cubical parabola) whose equation is

$$y^2 = x^3. \tag{2}$$

SOLUTION. Writing the equation in the form $f(x, y) = y^2 - x^3 = 0$, we have

$$\frac{\partial f}{\partial x} = -3x^2, \quad \frac{\partial f}{\partial y} = 2y.$$

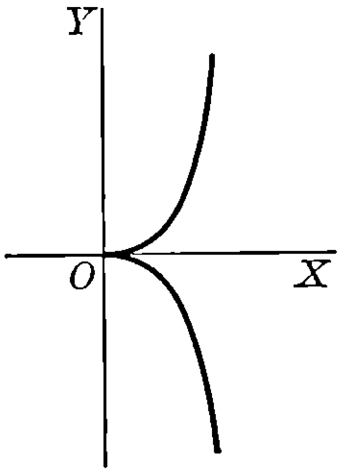

FIG. 89

The only pair of values (x, y) satisfying all three of the equations $f(x, y) = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ is evidently the pair $x = 0$, $y = 0$. Hence, by the theorem, the curve (2) can have no singular points except at the origin (0, 0).

That the origin is actually a singular point will appear from the theorem to be established presently, there being in fact a cusp at this point.

68. Algebraic curves. The general theorem of Art. 67 pertains to any curve $f(x, y) = 0$, but if we confine ourselves to the important special class known as *algebraic* curves, that is, those whose equations, at least after suitable reduction, contain only positive integral powers of x and y, we can go farther than the theorem states and assert that any point (x_1, y_1) at which both $\frac{\partial f_1}{\partial x_1} = 0$ and $\frac{\partial f_1}{\partial y_1} = 0$ is *necessarily* a singular point of the curve and we may furthermore readily find the equations of the two or more tangents which the curve possesses at this point. In order to show this, we first observe that the equation of every algebraic curve, in accordance with the foregoing definition, may be regarded as of the type form

$$\begin{aligned} f(x, y) = a_0 &+ (a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) \\ &+ (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \cdots = 0, \end{aligned} \tag{1}$$

the parentheses being inserted in order to group together for convenience the terms of the first degree, the terms of the second degree, third degree, etc. Moreover, we now observe that if (x_1, y_1) is a possible singular point of (1), as discovered by the theorem of Art. 67, we may at once take a new system of coördinate axes parallel to the old system and having their origin at (x_1, y_1)* in which case the equa-

* This is accomplished, as shown in analytic geometry (see, for example, the author's "Brief Course in Analytic Geometry," page 141), by merely substituting in (1) $x + x_1$ for x and $y + y_1$ for y.

tion takes a form similar to (1) except that, since the curve now passes through the origin, we shall have $a_0 = 0$. It suffices, then, to confine the attention to equations of the type

$$f(x, y) = (a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \quad = 0, \tag{2}$$

and the problem in hand is to ascertain the nature of the possible singular point of (2) at the origin (0, 0). In this connection we may now state and prove the following theorem:

Theorem. *If, in the algebraic equation*

$$f(x, y) = (a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \quad = 0,$$

the group of terms of the mth degree in x and y is the first group whose coefficients are not all equal to zero, then the curve having this equation has an m-tuple point at the origin, and the equations of the m tangents at this point may be obtained by equating this group of terms to zero; that is, they are expressed by the formula

$$a_mx^m + b_mx^{m-1}y + c_mx^{m-2}y^2 + \cdots + r_mxy^{m-1} + s_my^m = 0,$$

where not all the coefficients a_m, b_m, c_m, $\cdots$, r_m, s_m are equal to zero.

PROOF. From (2) we obtain

$$\frac{\partial f}{\partial x} = a_1 + (2\,a_2x + b_2y) + (3\,a_3x^2 + 2\,b_3xy + c_3y^2) + \cdots,$$

$$\frac{\partial f}{\partial y} = b_1 + (b_2x + 2\,c_2y) + (b_3x^2 + 2\,c_3xy + 3\,d_3y^2) +$$

Hence, by the theorem of Art. 64,

$$\frac{dy}{dx} = -\frac{a_1 + (2\,a_2x + b_2y) + (3\,a_3x^2 + 2\,b_3xy + c_3y^2) + \cdots}{b_1 + (b_2x + 2\,c_2y) + (b_3x^2 + 2\,c_3xy + 3\,d_3y^2) +}. \tag{3}$$

Suppose first that the first group of terms in (2) whose coefficients are not all equal to zero is the group $a_1x + b_1y$; that is, suppose that a_1 and b_1 are not both equal to zero. Then the value of y' at the origin (0, 0) may be determined from (3) and is seen to be

$$\frac{dy_1}{dx_1} = -\frac{a_1}{b_1}.$$

Hence, the origin is then an ordinary point and the equation of the single tangent at this point, as determined by the theorem of Art. 45, becomes

$$y - 0 = -\frac{a_1}{b_1}(x - 0), \text{ or } a_1x + b_1y = 0.$$

But this result is evidently the same as that obtained by simply

equating to zero the terms comprising the first group in (2), as stated in the theorem.

Suppose secondly that the second group of terms in (2) is the first one whose coefficients are not all equal to zero; that is, suppose that $a_1 = 0$ and $b_1 = 0$ but that a_2, b_2, and c_2 are not all equal to zero. Then the value of $\frac{dy}{dx}$ at the origin (0, 0) as determined by (3) is clearly of the form $\frac{0}{0}$ and hence is indeterminate, thus yielding no information. In order to remove this difficulty, let us suppose that the quadratic expression $a_2x^2 + b_2xy + c_2y^2$ has been factored (as is always possible) so that it is written in the form

$$a_2x^2 + b_2xy + c_2y^2 = c_2(y - m_1x)(y - m_2x), \tag{3}$$

where m_1 and m_2 are certain determinate numbers, real or imaginary.* Then (2) takes the form

$$f(x, y) = c_2(y - m_1x)(y - m_2x) + (a_3x^3 + b_3x^2y + \quad) + \quad = 0. \tag{4}$$

Now, any line through the origin, as $y = mx$, intersects the curve (4) in points whose abscissas are determined by the equation

$$c_2x^2(m - m_1)(m - m_2) + x^3(a_3 + b_3m + c_3m^2 + d_3m^3) + \quad = 0. \tag{5}$$

Two equal solutions of this equation are evidently $x = 0$, $x = 0$, but as we choose m closer and closer to either of the special values m_1 or m_2 the coefficient of x^2 in (5) approaches the limit zero; that is, a third point of intersection of the line $y = mx$ with the curve (4) then approaches the origin, since then (5) becomes an equation of higher order than the second, three of whose solutions are seen to be equal to zero. It follows that the two lines $y = m_1x$, $y = m_2x$ are each tangent to the curve at the origin. But the equations of these two lines may be written in the condensed form

$$a_2x^2 + b_2xy + c_2y^2 = 0,$$

as follows from (3). The theorem is therefore established for the case in which the second group of terms in (2) is the first one whose coefficients are not all equal to zero.

The reasoning just employed can evidently be extended to the case wherein the third group of terms in (2) is the first one whose coefficients are not all equal to zero. Similarly, the reasoning can be extended indefinitely, thus leading to the theorem as stated.

The proof is thus complete.

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* We here suppose that c_2 is not equal to zero. If it is, the reasoning which follows can be easily modified to care for such a case.

Example 1. Locate the singular points upon the algebraic curve

$$y^2 = x(x-1)^2 \qquad (1)$$

and determine for each the equations of the tangents.

SOLUTION. In the notation of Art. 68 we here have

$$f(x, y) = y^2 - x(x-1)^2$$

and hence

$$\frac{\partial f}{\partial x} = -3x^2 + 4x - 1, \qquad \frac{\partial f}{\partial y} = 2y.$$

The three equations $f(x, y) = 0, \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ are seen to be satisfied if, and only if, $x = 1$, $y = 0$. It follows by the theorem of Art. 67 that the only possible singular point upon the curve (1) is the point (1, 0).

Wishing now to apply the theorem of Art. 68 to (1), we first refer the curve to a new system of coördinate axes X', Y' which shall have the point under consideration, namely (1, 0), as origin and which shall be parallel respectively to the old axes X, Y To do this it suffices to replace the x and y of (1) by new variables x' and y' such that

$$x = x' + 1, \qquad y = y' + 0 = y' \qquad (2)$$

Equation (1) then becomes $y'^2 = (x' + 1)x'^2$, which may be written

$$y'^2 - x'^2 - x'^3 = 0. \qquad (3)$$

Inasmuch as the singular point under consideration is now at the origin, we may at once apply the theorem of the present article to (3), thus inferring that the curve in question has a node at the (new) origin and that the equations of the two tangents at this point are, as referred to the new axes, $y'^2 - x'^2 = 0$; that is,

$$y' = x' \quad \text{and} \quad y' = -x'.$$

These equations, when referred again to the old axes by means of the equations of transformation (2), are seen to become

$$y = x - 1 \quad \text{and} \quad y = -x + 1.$$

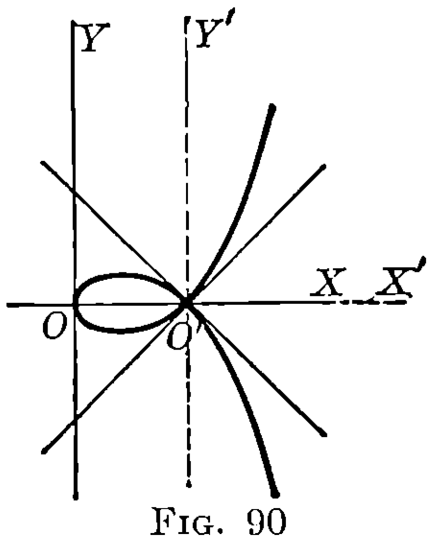

FIG. 90

In summary, we conclude that the curve (1) has one and only one singular point, this being a node situated at (1, 0) and having for its accompanying tangents the lines $y = x - 1$ and $y = -x + 1$.

Example 2. Show that the curve $3x^2 + 4x^3 - y^3 = 0$ has a cusp of the first kind at the origin.

SOLUTION. In order to determine the nature of the curve at the origin (0, 0) (which is evidently a point upon the curve) it suffices to apply directly the theorem of the present article. It thus appears that the origin is a double point having its two tangents determined by the equation $3x^2 = 0$, or simply $x^2 = 0$. The two tangents are therefore coincident, each coinciding with the

line $x = 0$, or Y-axis. By definition (Art. 66) the origin is therefore a cusp. That it is a cusp of the *first* kind may now be shown as follows:

Writing the given equation in the form

$$y^3 = 3x^2 + 4x^3,$$

it appears (1) that when x is very small and positive (though not actually zero) the corresponding values of y are positive; (2) that when x is sufficiently small and negative (though not zero) the corresponding values of y are again positive since $4x^3$, which is then negative, eventually becomes smaller numerically than $3x^2$, which is always positive. Thus, the given equation when considered for very small values of x of either sign, gives positive values of y. This fact, when interpreted geometrically, indicates that the curve, when considered sufficiently near the origin, exists on either side of the common tangent, $x = 0$ and lies above the X-axis. The cusp, therefore, is of the *first* kind.

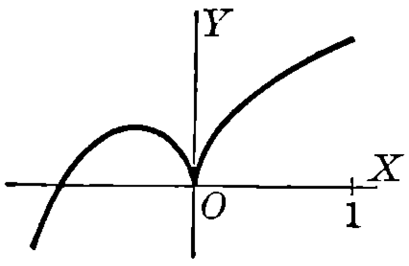

FIG. 91

EXERCISES

1. Show that the curve $y = \sin x$ can have no singular point. Do the same for each of the curves $y = \cos x$, $y^2 = 4x$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y = e^x$, $y = \log x$.

2. Establish the following statements: (*a*) The curve $x^2 - 4y^2 + x^3 - y^3 = 0$ has a node at the origin, the equations of the two tangents at this point being $x = \pm 2y$.

(*b*) The curve $x^2 - y^3 + y^4 = 0$ has a cusp at the origin, the two common tangents at this point being $x^2 = 0$.

(*c*) The curve $1 - 2x + y + 2y^2 + x^2 + y^3 = 0$ has a node at the point $(1, -1)$, the equations of the two tangents at this point being $x + y = 0$ and $x - y = 2$.

3. Determine the location and nature (whether node or cusp) of each of the singular points (if any) upon each of the following algebraic curves, stating also whether the point in question is a double point, triple point, etc. Also determine in each instance the equations of the tangents (real or imaginary) at the point in question. Do not attempt to sketch the complete curve, as this may be undertaken to better advantage after later parts of this chapter have also been considered.

(*a*) $x^2 - y^2 - x^4 = 0$;

(*b*) $(y - 1)^2 = 2(x + 1)^3$;

(*c*) $x^3 + y^3 - 3xy = 0$;

(*d*) $x^3 - xy^2 + y^4 = 0$;

(*e*) $x^4 - y^4 + x^3y^3 = 0$;

(*f*) $x^3 + x^2 + y^2 = 0$;

(*g*) $x^2 + xy + y^2 + x^3 - y^3 = 0$;

(*h*) $x(x - 1)^2 = y^4$;

(*i*) $y^3 = 2ax^2 - x^3$;

(*j*) $xy = 4$.

4. Show that the origin is an ordinary point upon the cubic $x^3 + y^3 - x + y = 0$, and hence write, by use of the theorem of Art. 68, the equation of the single tangent at this point.

5. Show that the origin is an ordinary point upon the cubic $x^3 + y^3 + 3x^2 - 18xy + 6y^2 - 6x + 21y = 0$, but that there exists a cusp at the point (2, 1).

6. Show that the conchoid $x^2y^2 = (a + y)^2(b^2 - y^2)$ has a node at the point $(0, -a)$ provided that $b > a$.

7. By means of the theorem of Art. 68 write the equation of the tangent to the conic (ellipse) $5x^2 - 5xy + 3y^2 - 4x + y = 0$ at the origin.

8. By means of the theorem of Art. 68 obtain the equation of the tangent to the conic (hyperbola) $3x^2 + 4xy - 5y^2 + 6x - 8y - 33 = 0$ at the point (3, 2).

HINT. First refer the curve to a new set of axes having (3, 2) as origin (see Example 1, Art. 68), then proceed as in Ex. 7.

9. By proceeding as in Exs. 7 and 8, prove the following general theorem: "Having given the equation of any conic

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

the equation of the tangent at any point (x_1, y_1) upon the curve is given by the formula

$$Ax_1x + H(y_1x + x_1y) + By_1y + G(x_1 + x) + F(y_1 + y) + C = 0."$$

10. Show that the curve $y^2 + x^3 + y^3 = 0$ has a cusp of the first kind at the origin.

11. Show that the curve $2x + 3y - x^2 + 3y^2 + y^3 = 0$ has a cusp of the first kind at the point $(1, -1)$.

12. Show that the curve $(y - x^2)^2 = x^5$ has a cusp of the second kind at the origin.

HINT. By solving the equation for y show analytically that the curve, when considered in the neighborhood of the origin, exists on one side only of the cuspidal tangent.

13. Prove that the curve $(y - x^2)^2 = x^n$ has a cusp at the origin and that it is of the first or second kind according as $n < 4$ or $n > 4$.

14. Show that the curve $x^4 + x^3 - 3xy^2 = 0$ has a triple point at the origin. Show also that the three tangents at this point divide the whole angle around the origin into six equal parts.

69. Asymptotes. Let AB be any curve, or branch of a curve, which extends indefinitely to distant portions of the plane, or as commonly stated, "extends to infinity." As the point of contact P of a tangent to such a curve is chosen farther and farther away the accompanying tangent ST may approach a definite limiting position MN In this case MN is called an **asymptote** to the curve AB.

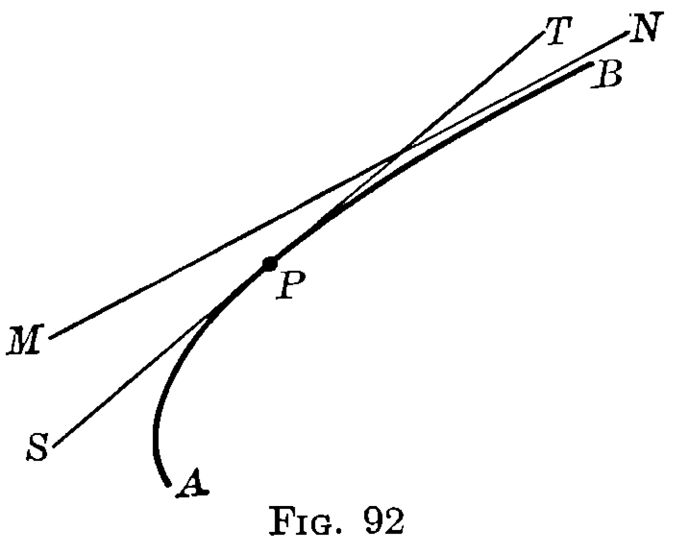

FIG. 92

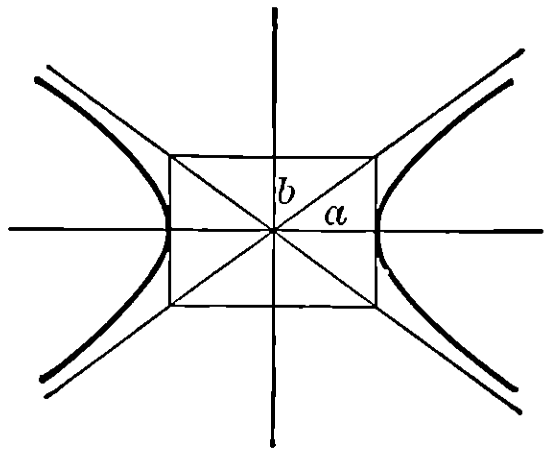

FIG. 93

Thus, it is shown in analytic geometry that the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two asymptotes situated as in Fig. 93 having as their equations

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0.$$

On the other hand, the parabola has no asymptotes since, as the point of contact of a tangent line recedes indefinitely, the accompanying tangent approaches no limiting position. Again, in the case of the ellipse there can be no asymptotes inasmuch as the curve has no branch of indefinite extent.

We proceed to state and prove the following theorem in accordance with which one may frequently detect the presence of asymptotes to an algebraic curve and find the equations of such lines.

Theorem I. *The line*

$$y = mx + k \tag{1}$$

will be an asymptote to an algebraic curve whenever the coefficients m and k are so determined that, when mx + k has been substituted for y in the equation of the curve and the resulting equation arranged according to descending powers of x, the coefficients of the two highest powers of x are each equal to zero.

PROOF. It is first to be remarked that, from the definition of an asymptote in Art. 69, it readily appears that an asymptote to any curve may, if desired, be regarded as the limiting position (assumed to exist) of a secant line as it is moved about in such a manner that two of its points of intersection with the curve recede indefinitely along the curve, or as commonly stated, " go to infinity." It is from this standpoint that we shall find it advantageous to regard an asymptote in the present proof.

The result of substituting $mx + k$ for y in (1), Art. 68, and subsequently arranging the result in descending powers of x is an equation of the type form

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \quad + A_{n-1}x + A_n = 0, \tag{2}$$

where the coefficients A_0, A_1, A_2, involve m and k. As regards such an equation as (2), it is not only shown in algebra that it has n roots (real or imaginary, distinct or coincident in part or in whole)

but it is also shown that the smaller A_0 is taken the larger one of the n roots will become in absolute value. In fact, it is shown that as A_0 is allowed to approach zero as a limit one of the roots of (2) becomes indefinitely large in absolute value, or as commonly stated, "goes to infinity." Similarly, it is shown that if both A_0 and A_1 are allowed to approach zero, two of the n roots of (2) go to infinity. Hence, recalling that the roots of (2) when considered geometrically are the abscissas of the points of intersection of the line (1) with the given curve, it follows that if m and k are so determined that $A_0 = 0$ and $A_1 = 0$, the resulting line (1) will have two of its intersection points with the curve lying "at infinity," thus making such a line an asymptote.

The proof is thus complete.

Remark. In the case of an asymptote which is parallel to the Y-axis the slope m must be regarded as $\pm\infty$, hence such an asymptote does not have its equation of the type form (1) but rather of the type form $x = k$, where k is some constant. For this reason the foregoing theorem cannot be expected to detect asymptotes of this character. Asymptotes parallel to the Y-axis may, however, usually be detected independently through the following theorem.

Theorem II. *Having arranged the equation of a given algebraic curve of the nth degree in descending powers of y, thus causing it to assume the form*

$$ay^n + (bx + c)y^{n-1} + (dx^2 + ex + f)y^{n-2} + \qquad = 0, \tag{3}$$

if the coefficient a is equal to zero while the coefficient b is not equal to zero, then the line

$$bx + c = 0, \qquad \text{or} \qquad x = -\frac{c}{b}, \tag{4}$$

(which is parallel to the Y-axis) will form an asymptote to the given curve.

PROOF. Reasoning as in the proof of Theorem I, it appears from the hypothesis $a = 0$ that every line of the type form $x = k$ intersects the curve (3) in at least one point at infinity. If in particular we have $x = -\frac{c}{b}$, which is equivalent to (4), the coefficients of both y^n and y^{n-1} in (3) will be equal to zero, so that this particular line will intersect the curve in two points at infinity, thus causing it to become an asymptote.

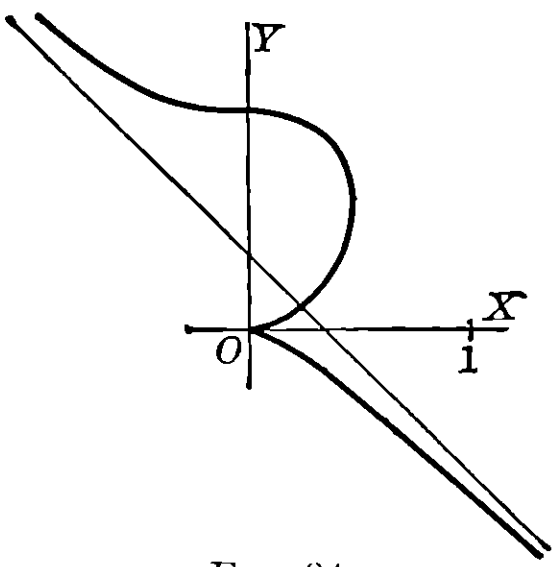

FIG. 94

Example 1. Test the following curve for asymptotes:

$$x^3 + y^3 = y^2. \qquad (5)$$

SOLUTION. First arrange the equation in descending powers of y. Thus (5) takes the form

$$y^3 - y^2 + x^3 = 0. \qquad (6)$$

Replacing y by $mx+k$, as directed by Theorem I, and arranging the result in descending powers of x, (6) becomes

$$(mx + k)^3 - (mx + k)^2 + x^3$$
$$= (m^3 + 1)x^3 + (3\,m^2k - m^2)x^2 + (3\,mk^2 - 2\,mk)x + k^3 - k^2 = 0. \qquad (7)$$

Noting the coefficients of the two highest powers of x and equating these to zero, as also directed by Theorem I, we obtain the following two equations for determining values of m and k that will cause (1) to be an asymptote:

$$m^3 + 1 = 0, \qquad 3\,m^2k - m^2 = 0.$$

From these equations we obtain at once $m = -1$, $k = \frac{1}{3}$. Hence, the line $y = -x + \frac{1}{3}$, or $3\,x + 3\,y = 1$, is an asymptote to the given curve (5).

It remains to see whether Theorem II may also be applied, thus discovering one or more asymptotes that are parallel to the Y-axis. Theorem II, however, cannot be applied in the present instance inasmuch as the coefficient of the highest power of y in (6) is not equal to zero but to $+1$. There are, in fact, no asymptotes parallel to the Y-axis.

Example 2. Test the following curve for asymptotes:

$$y^2 = \frac{x^3}{2\,a - x}.$$

SOLUTION. Clearing of fractions and rearranging, the equation becomes

$$y^2(x - 2\,a) + x^3 = 0. \qquad (8)$$

Placing $y = mx + k$ in this equation and arranging the result in descending powers of x, we obtain

$$(m^2 + 1)x^3 + 2\,m(k - am)x^2 + k(k - 4\,am)x - 2\,ak^2 = 0.$$

Here the coefficient of the highest power of x is $m^2 + 1$, and this evidently cannot be equal to zero for any real values of m. Hence Theorem I does not apply. There are, in fact, no asymptotes in the present instance except such as may be detected as follows by use of Theorem II.

Writing (8) in the form

$$0 \cdot y^3 + (x - 2\,a)y^2 + x^3 = 0,$$

it appears immediately that Theorem II applies and that therefore the line $x - 2\,a = 0$, or $x = 2\,a$, is an asymptote parallel to the Y-axis.

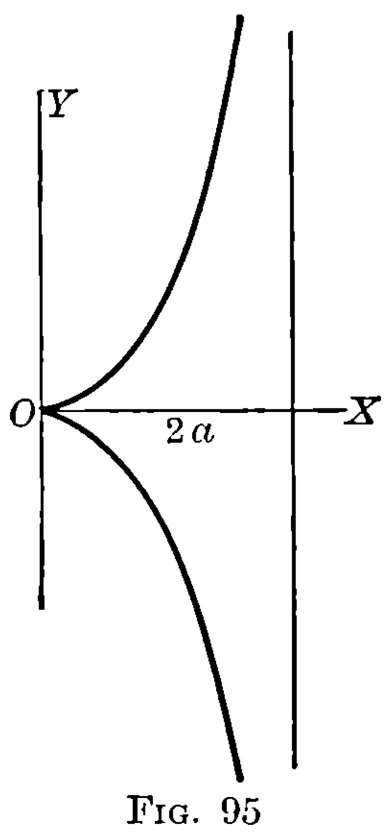

FIG. 95

70. Exceptional cases. In the ordinary applications of Theorem I, Art. 69, to an equation of the nth degree the result of substituting $mx + k$ for y, as directed by the theorem, will be an equation in x of the form

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \quad + A_{n-1}x + A_n = 0,$$

where A_0 contains m only while A_1 contains both m and k but contains k to no higher degree than the first, as illustrated in (7) of Art. 69. The two equations $A_0 = 0$, $A_1 = 0$ therefore ordinarily determine uniquely certain values of m and a *single* value of k corresponding to each. It may happen, however, in exceptional cases that the values of m which render $A_0 = 0$ cause A_1 to be equal to zero identically, that is, for *all* values whatsoever of k; or again it may happen that A_1 is identically zero no matter what values be given to both m and k. In such cases the two equations to be used for determining m and k become $A_0 = 0$ and $A_2 = 0$ instead of $A_0 = 0$ and $A_1 = 0$, it being now assumed that A_2 is not identically equal to zero, in which case the two equations become $A_0 = 0$ and $A_3 = 0$, etc. Theorem I as worded in Art. 69 remains true in all cases, though the proof of it as there given needs some modification in the exceptional cases just mentioned.

Thus, if $mx + k$ be substituted for y in the equation

$$x^3 + 2\,x^2y + y^2x + y = 0, \tag{1}$$

the result, when arranged in descending powers of x, may be written

$$(1 + m)^2x^3 + 2\,k(1 + m)x^2 + (k^2 + m)x + k = 0.$$

Here

$$A_0 = (1 + m)^2, \quad A_1 = 2\,k(1 + m), \quad A_2 = k^2 + m, \quad A_3 = k.$$

From the equation $A_0 = 0$ we obtain $m = -1$, and for this value of m the equation $A_1 = 0$ is seen to be true identically for all values of k, while the equation $A_2 = 0$ becomes $k^2 - 1 = 0$, the roots of which are $k = +1$ and $k = -1$.

The parallel lines $y = -x + 1$ and $y = -x - 1$ are therefore asymptotes to the curve whose equation is (1).

Likewise, in applying Theorem II of Art. 69 it may happen that a, b, and c are each equal to zero. In this case there will be, in general, two asymptotes parallel to the Y-axis, and their equations, in case d is not also equal to zero, will be given by the equation $dx^2 + ex + f = 0$ instead of $bx + c = 0$. The various other exceptional cases and the method of disposing of each may now be readily supplied by the student from analogy with the preceding remarks.

For example, by writing the equation

$$(x^2 - 1)y + x^3 = 0 \qquad (2)$$

in the form

$$0 \cdot y^3 + 0 \quad y^2 + (x^2 - 1)y + x^3 = 0$$

we may assert at once that the curve having (2) as its equation has two asymptotes which are parallel to the Y-axis and whose equations are $x^2 - 1 = 0$; that is, $x = 1$ and $x = - 1$.

EXERCISES

1. Apply Theorem I, Art. 69, to each of the following curves to determine the equations of such of its asymptotes as are not parallel to the Y-axis:

(*a*) the hyperbola $4x^2 - 9y^2 = 36$; *Ans.* $2x = \pm 3y$.

(*b*) the hyperbola $x^2 + xy - 6y^2 - 30y + 36 = 0$; *Ans.* $x + 3y = - 6$, $x - 2y = 6$.

(*c*) $x^2 - 2xy - 3y^2 + 2x - 2y - 4 = 0$; *Ans.* $3y - x = 1$, $x + y = - 1$.

(*d*) $x^3 + x^2y + y^2 + xy = 0$; *Ans.* $x + y = 0$.

(*e*) $x^3 - y^2 - x^2 - xy + y = 0$;

(*f*) $y^3 + 3xy^2 + 2x^2y + x^2 = 0$;

(*g*) $x^3 + 4x^2y + 4xy^2 + 8y = 0$. HINT. See Art. 70.

2. Apply Theorem II, Art. 69, to determine the equations of asymptotes parallel to the Y-axis for each of the following curves:

(*a*) $(x - 1)y^2 + (x^2 + 1)y + x^3 = 0$; *Ans.* $x = 1$.

(*b*) $x^3 + x^2y + xy^2 - xy + 3y^2 + y = 0$; *Ans.* $x = - 3$.

(*c*) $x^4 + x^2y^2 + xy^3 + 2y^3 = 0$;

(*d*) $2x^3 + x^2y - 3xy + 2y = 0$. HINT. See Art. 70.

3. Examine the following curves for asymptotes:

(*a*) $xy = 4$;

(*b*) $x^2 - xy + y^2 + 3x = 0$;

(*c*) $y^2(x^2 + 1) = x^2(x^2 - 1)$;

(*d*) $x^3 + y^3 = a^3$;

(*e*) $y^2(x - 2a) - x^3 + a^3 = 0$;

(*f*) $x^2y^2 = a^2(x^2 + y^2)$.

71. Curve tracing. The familiar considerations of elementary analytic geometry relative to curve tracing, when supplemented by the results attainable in the calculus such as indicated in this chapter, enable one ordinarily to arrive quickly at the essential features of any algebraic curve whose equation is given. While no fixed rules can be stated, one may usually proceed as follows:

(1) *Determine* (*so far as can be done without too much labor*) *the possible region, or regions, within which the curve must lie.* To do this, recall that the curve is restricted to those points (x, y) for which both x and y are real numbers.

(2) *Determine if possible the points, if any, at which the curve intersects the axes of coördinates.*

(3) *Determine whether the curve is symmetrical with respect (a) to the X-axis; (b) to the Y-axis; (c) to the origin.* There will be symmetry with respect to the X-axis if, and only if, the equation remains unchanged when y is replaced by $-y$; similarly, there will be symmetry with respect to the Y-axis in case x may be changed to $-x$. There will be symmetry with respect to the origin whenever both these changes can be made in the equation.

(4) *Determine the maximum and minimum points, if any.* (Art. 37)

(5) *Determine (unless too much labor is involved) the points of inflection, if any.* (Art. 41)

(6) *Determine the position and character of the singular points, if any.* (Arts. 67, 68)

(7) *Determine the position of the asymptotes, if any.* (Art. 69)

Example. Trace the curve whose equation is

$$y^2 = \frac{x^3}{2a - x}.$$

SOLUTION. (1) The curve is confined to regions in which x is less than $2a$, for if x is greater than $2a$, the sign of y^2 is evidently negative, in which case y is imaginary.

(2) If y has the special value zero, x is seen to be necessarily equal to zero also, while if x has the value zero, y is necessarily zero. Hence the curve does not cross either the X-axis or the Y-axis except at the origin.

(3) The curve is symmetrical with respect to the X-axis since the equation remains unchanged when $-y$ is substituted for y. The curve is not symmetric with respect to the Y-axis since $-x$ is not interchangeable with x.

(4) Upon taking the derivative of both members of the equation and solving the resulting equation for $\frac{dy}{dx}$ we obtain, after simple reductions,

$$\frac{dy}{dx} = \frac{(3a - x)x^2}{(2a - x)^2 y}.$$

The only values at which $\frac{dy}{dx}$ can be equal to zero, thus leading to a possible maximum or minimum point, are therefore $x = 3a$ and $x = 0$. Of these, $x = 3a$ is inadmissible because, as stated in (1), x is confined to values less than $2a$, and hence must be less than $3a$. Moreover, if $x = 0$ we see from the equation of the curve that y is also equal to zero. Hence, when $x = 0$ the foregoing expression for $\frac{dy}{dx}$ takes the indeterminate form $\frac{0}{0}$ from which no

conclusion can be drawn (Art. 6). Rather than undertake to remove this uncertainty immediately by further consideration of $\frac{dy}{dx}$ it is advisable, since difficulty is present, to pass at once to a determination of the other important features of the curve. In fact, these when ascertained will automatically remove the present doubt as to maxima and minima points.

(5) The consideration of points of inflection, being dependent upon the expression for $\frac{dy}{dx}$, is likewise best omitted in the present case.

(6) Clearing the given equation of fractions and rearranging, it may be written

$$f(x, y) = 2\,ay^2 - x^3 - xy^2 = 0.$$

Therefore

$$\frac{\partial f}{\partial x} = -3\,x^2 - y^2, \qquad \frac{\partial f}{\partial y} = 4\,ay - 2\,xy.$$

The equations $f(x, y) = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ are evidently satisfied if, and only if, $x = 0$ and $y = 0$. Hence, by Art. 68, the curve has the origin (0, 0) as a singular point, and no other such points can be present. Moreover, this singular point at (0, 0) is seen, by the theorem of Art. 68, to be a cusp having the line $y = 0$ as double tangent.

(7) As shown in Example 2, Art. 69, this curve has the line $x = 2\,a$ as an asymptote.

From the conclusions arrived at in (1), (2), (3), (6), and (7) we may now conclude that the curve has the form indicated in Fig. 95, page 144. This is the curve of great historic interest known as the *cissoid.**

EXERCISES

1. Trace the curves whose equations occur in Ex. 3, page 140.

2. Trace the set of curves:

$$y = x^5,\ y = x^5 - x,\ y = x^5 - x^2,\ y = x^5 - x^3,\ y = x^5 - x^4.$$

3. Trace the set:

$$x + y = a,\ x^2 + y^2 = a^2,\ x^3 + y^3 = a^3,\ x^1 + y^4 = a^4.$$

4. Trace the curve

$$x^3 - 3\,xy^2 = 8.$$

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* See, for example, the historical note on page 169 of the author's "Brief Course in Analytic Geometry."

CHAPTER IX

LAW OF THE MEAN. INDETERMINATE FORMS

72. The law of the mean. In Fig. 96 let AB be the arc of a continuous curve at every point of which a tangent line of finite slope can be drawn. Then, if any two points P_1, P_2, be taken on AB and the straight line P_1P_2 be drawn, it is evident that a point Q may be located upon AB such that the tangent to the curve at Q will be parallel to the direction of P_1P_2, whatever this direction may be. In cases such as indicated in Fig. 97 it is possible, in fact, to locate more than one such point Q, but in all cases it is possible to locate at least one such intermediate point as soon as the two points P_1 and P_2 are given.

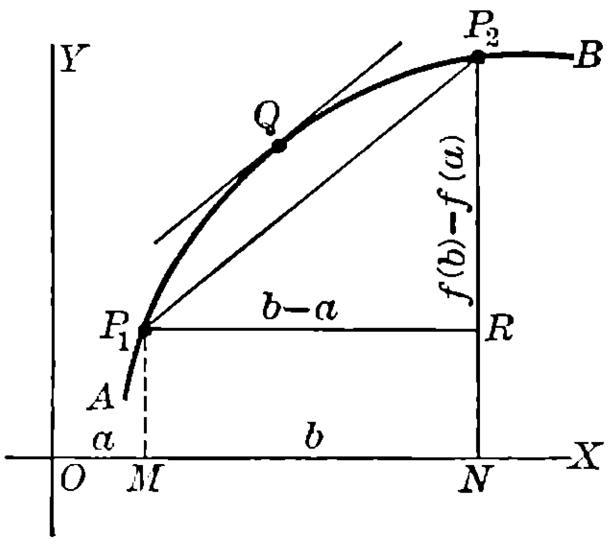

Fig. 96

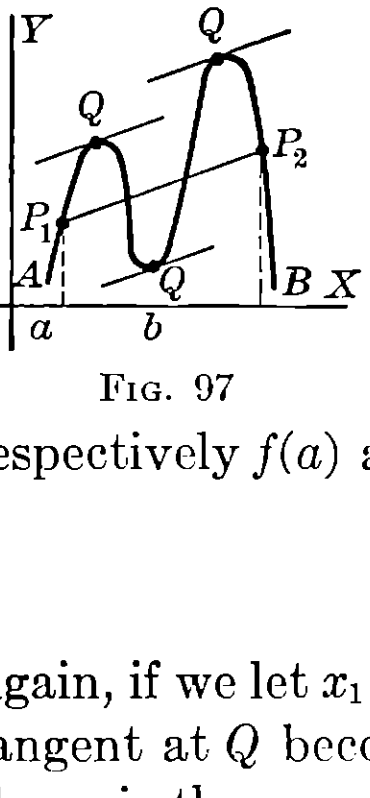

Fig. 97

Let us now formulate this fact analytically. For this purpose suppose that the abscissas of P_1 and P_2 are respectively a and b, in which case the ordinates MP_1 and NP_2 of the same points become respectively $f(a)$ and $f(b)$. Hence the slope of the line P_1P_2 becomes

$$\frac{RP_2}{P_1R} = \frac{f(b) - f(a)}{b - a}.$$

Again, if we let x_1 represent the abscissa of the point Q, the slope of the tangent at Q becomes, by Art. 9, $f'(x_1)$. The geometric fact stated above is then equivalent to the equation

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

or

$$f(b) = f(a) + (b - a)f'(x_1),$$

it being understood that x_1 is some determinate value of x lying between a and b. This relation is known as the **law of the mean.** Its importance lies chiefly in the fact that it may be used to advantage in proving various theorems in the calculus, as will appear presently.

73. Definitions. From Theorem III of Art. 5 * it follows that the limit approached by the quotient of two functions is equal ordinarily to the limit of the numerator, considered separately, divided by the limit of the denominator, likewise considered separately; that is, one may write, unless $F(a) = 0$,

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} F(x)} = \frac{f(a)}{F(a)}. \tag{1}$$

Thus,

$$\lim_{x \to 4} \frac{x^2+1}{x+1} = \frac{4^2+1}{4+1} = \frac{17}{5}; \ \lim_{x \to e} \frac{\log x}{x} = \frac{\log e}{e} = \frac{1}{e}; \ \lim_{x \to \pi} \frac{\sin x}{x} = \frac{\sin \pi}{\pi} = \frac{0}{\pi} = 0.$$

Formula (1), however, ceases to apply or indeed to have any definite significance in case $F(a)$ is equal to zero, since division by zero is an impossible operation in mathematics. In this exceptional case the limit which is being considered in the first member of (1) may still exist and have definite significance, provided that we have at the same time $f(a) = 0$. To determine the limit (1) in such cases it is usually necessary to use some special process.

Thus, suppose it is desired to evaluate the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}. \tag{2}$$

Here the direct application of (1) would give

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0},$$

which is meaningless. The limit (2), however, exists and may be obtained independently by first observing that, by elementary algebra, we have

$$\frac{x^2 - 1}{x - 1} = x + 1.$$

Thus, we may write

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

Similarly, the application of (1) to the determination of $\lim_{x \to 0} \frac{\sin x}{x}$ gives $\frac{0}{0}$. However, this limit is known to exist and to have the value 1 provided that x is measured in radians. (See page 360.)

* The student is advised to review at this point Arts. 4–6.

Whenever in the use of (1) we have both $f(a) = 0$ and $F(a) = 0$, the limit in question is said to assume the **indeterminate form** $\frac{0}{0}$.

Again, relation (1) cannot be used to obtain decisive results in case both $f(x)$ and $F(x)$ become infinite as x approaches a. Nevertheless, the limit in question may exist in such a case and be actually determinate by other special methods of procedure.

Thus, (1) is not immediately applicable to the determination of the limit

$$\lim_{x \to +\infty} \frac{3x^2 + 2x - 1}{x^2 - 3x + 3},$$

since both the numerator and denominator become infinite as $x \to +\infty$. However, the limit in question exists and may be evaluated by first dividing both numerator and denominator of the fraction by x^2 and then using (1). Thus,

$$\lim_{x \to +\infty} \frac{3x^2 + 2x - 1}{x^2 - 3x + 3} = \lim_{x \to +\infty} \frac{3 + \frac{2}{x} - \frac{1}{x^2}}{1 - \frac{3}{x} + \frac{3}{x^2}} = \frac{3 + 0 - 0}{1 - 0 + 0} = 3.$$

Whenever in the use of (1) both $f(x)$ and $F(x)$ become infinite as x approaches a, the limit in question is said to assume the **indeterminate form** $\frac{\infty}{\infty}$.

74. Evaluation of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$. Whenever a limit assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, it may, in general, be evaluated by a uniform method based upon the calculus, thus avoiding special individual devices such as were used in the illustrations in Art. 73. In this connection we have, in fact, the following general theorem:

Theorem. *In case the limit* $\lim_{x \to a} \frac{f(x)}{F(x)}$ *assumes the indeterminate form* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$, *then*

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)},$$

provided the latter limit exists.

PROOF. We shall here prove the theorem only for the case in which the limit in question assumes the indeterminate form $\frac{0}{0}$, the case $\frac{\infty}{\infty}$ presenting greater difficulties and hence being better undertaken in a more advanced treatment of the calculus.

By the law of the mean (Art. 72) as applied to the functions $f(x)$ and $F(x)$ throughout the interval from a to x, we may write

$$f(x) = f(a) + (x - a)f'(x_1),$$

$$F(x) = F(a) + (x - a)F'(x_2),$$

where x_1 and x_2 are certain values each lying between a and x. But our present hypotheses require that $f(a) = 0$ and $F(a) = 0$ since we are dealing with the indeterminate form $\frac{0}{0}$. Thus, from the two foregoing equations we obtain

$$\frac{f(x)}{F(x)} = \frac{(x - a)f'(x_1)}{(x - a)F'(x_2)} = \frac{f'(x_1)}{F'(x_2)}.$$

If in this equation we now allow x to approach a, noting that at the same time x_1 and x_2 must necessarily do the same, we arrive at the following equation, it being assumed that the limit appearing in its second member exists:

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)}.$$

The proof is thus complete for the case $\frac{0}{0}$.

Note 1. In applying the foregoing theorem it may happen that the fraction $\frac{f'(x)}{F'(x)}$ itself assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when the $\lim_{x \to a}$ is taken. In such a case, applying the theorem a second time, we differentiate the numerator and the denominator again, obtaining $\frac{f''(x)}{F''(x)}$, and we proceed to investigate $\lim_{x \to a} \frac{f''(x)}{F''(x)}$. If this is still indeterminate, we investigate $\lim_{x \to a} \frac{f'''(x)}{F'''(x)}$, etc.

Note 2. Although the theorem as stated has reference only to a limit of the type $\lim_{x \to a}$ where a is some definite number, it holds true equally well for limits of the type $\lim_{x \to +\infty}$ and $\lim_{x \to -\infty}$ (see Art. 4). Proof of this fact is omitted, however, for the sake of brevity.

Example 1. Evaluate the limit $\lim\limits_{x \to 1} \dfrac{\log x}{x - 1}$.

SOLUTION. Since both $\log x$ and $x - 1$ become equal to zero when $x = 1$, the limit in question assumes the indeterminate form $\dfrac{0}{0}$. We may therefore apply the foregoing theorem as follows, thus showing that the desired limit is equal to 1:

$$\lim_{x \to 1} \frac{\log x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx} \log x}{\dfrac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{\dfrac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1.$$

Example 2. Evaluate the limit $\lim\limits_{x \to +\infty} \dfrac{x^2 + x - 1}{e^x + e^{-x}}$.

SOLUTION. Since $\lim\limits_{x \to +\infty} (x^2 + x - 1) = +\infty$ and $\lim\limits_{x \to +\infty} (e^x + e^{-x}) = +\infty$, the limit in question assumes the indeterminate form $\dfrac{\infty}{\infty}$. We therefore proceed as follows:

$$\lim_{x \to +\infty} \frac{x^2 + x - 1}{e^x + e^{-x}} = \lim_{x \to +\infty} \frac{\dfrac{d}{dx}(x^2 + x - 1)}{\dfrac{d}{dx}(e^x + e^{-x})} = \lim_{x \to +\infty} \frac{2x + 1}{e^x - e^{-x}}.$$

The last limit here appearing still assumes the indeterminate form $\dfrac{\infty}{\infty}$, hence we apply the same process again. Thus,

$$\lim_{x \to +\infty} \frac{2x + 1}{e^x - e^{-x}} = \lim_{x \to +\infty} \frac{\dfrac{d}{dx}(2x + 1)}{\dfrac{d}{dx}(e^x - e^{-x})} = \lim_{x \to +\infty} \frac{2}{e^x + e^{-x}} = 0.$$

The limit in question therefore has the value zero.

EXERCISES

1. Establish each of the following equations by means of the theorem of Art. 74, checking the answer for the first four by the elementary methods illustrated in Art. 6 and Exs. 8, 9, page 9.

(a) $\lim\limits_{x \to -2} \dfrac{x^2 - 4}{x + 2} = -4;$

(b) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x^2 + x - 2} = \dfrac{2}{3};$

(c) $\lim\limits_{x \to 2} \dfrac{x^2 - 4x + 4}{x^3 - 3x^2 + 4} = \dfrac{1}{3};$

(d) $\lim\limits_{x \to +\infty} \dfrac{x^3 + 2x^2 - x + 3}{2x^3 + x^2 - 1} = \dfrac{1}{2};$

(e) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}} = -1;$

(f) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x} = 2.$

2. Evaluate each of the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\log x}{\dfrac{1}{x}}$;

(b) $\lim\limits_{x\to 0} \dfrac{\log x}{\cot x}$;

(c) $\lim\limits_{x\to +\infty} \dfrac{x}{e^x}$;

(d) $\lim\limits_{x\to +\infty} \dfrac{x^n}{e^x}, n > 0$;

(e) $\lim\limits_{x\to +\infty} \dfrac{\log x}{x}$;

(f) $\lim\limits_{x\to +\infty} \dfrac{(\log x)^n}{x}, n > 0$;

(g) $\lim\limits_{x\to 1} \dfrac{\log (2x^2 - 1)}{\tan (x-1)}$;

(h) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$;

(i) $\lim\limits_{x\to 0} \dfrac{\text{arc}\tan x - x}{x^3}$;

(j) $\lim\limits_{x\to 1} \dfrac{\log x}{\sqrt{1-x}}$.

75. The indeterminate form $0 \cdot \infty$. Let the product of two functions be given, as $f(x) \cdot F(x)$, and suppose that as x approaches a one function approaches zero while the other increases indefinitely in numerical value. The product is then said to assume the indeterminate form $0 \cdot \infty$ as x approaches a. Such a form may, however, be converted into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and thus treated by the theorem of Art. 74, for we may write

$$f(x) \cdot F(x) = \frac{f(x)}{\dfrac{1}{F(x)}},$$

and the quotient thus obtained assumes either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as x approaches a.

Example. Evaluate $\lim\limits_{x\to 0} \tan x \cdot \log \sin x$.

SOLUTION. Having noted that the form $0 \cdot \infty$ is here presented, we write

$$\lim_{x\to 0} \tan x \cdot \log \sin x = \lim_{x\to 0} \frac{\log \sin x}{\dfrac{1}{\tan x}} = \lim_{x\to 0} \frac{\log \sin x}{\cot x} = \frac{\infty}{\infty}.$$

Applying the theorem of Art. 74, we have

$$\lim_{x\to 0} \frac{\log \sin x}{\cot x} = \lim_{x\to 0} \frac{\dfrac{d}{dx}\log \sin x}{\dfrac{d}{dx}\cot x} = \lim_{x\to 0} \frac{\dfrac{\cos x}{\sin x}}{-\csc^2 x} = -\lim_{x\to 0} \frac{\dfrac{\cos x}{\sin x}}{\dfrac{1}{\sin^2 x}}$$

$$= -\lim_{x\to 0} \sin x \cdot \cos x = -0 \cdot 1 = 0.$$

The value of the limit in question is therefore zero.

76. The indeterminate form $\infty - \infty$. Let the difference $f(x) - F(x)$ of two functions be given and let us suppose that $\lim_{x \to a} f(x) = +\infty$ and $\lim_{x \to a} F(x) = +\infty$. Then $f(x) - F(x)$ is said to assume the indeterminate form $\infty - \infty$ as x approaches a. Such a form, like the form $0 \cdot \infty$ mentioned in Art. 75, may be converted into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example. Evaluate $\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{1}{x - 1} \right)$.

SOLUTION. Having noted that this limit assumes the form $\infty - \infty$, we may proceed as follows:

$$\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{1}{x - 1} \right) = \lim_{x \to 1} \left[\frac{x - 1 - \log x}{(x - 1) \log x} \right] = \frac{0}{0}.$$

The theorem of Art. 74 may now be applied, the result being found directly to be $\frac{1}{2}$.

77. The indeterminate forms 0^0, ∞^0, 1^∞. An expression of the form $[f(x)]^{F(x)}$, where $f(x)$ and $F(x)$ are two given functions, becomes indeterminate as x approaches a in case it assumes either of the forms 0^0, ∞^0, or 1^∞ Such indeterminates may be made to depend upon one or the other of the forms considered in Arts. 74, 75, and hence may usually be evaluated in the manner illustrated in the following example.

Example. Evaluate $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

SOLUTION. First observe that this limit assumes the indeterminate form 1^∞. In order to evaluate, let

$$y = (1 + x)^{\frac{1}{x}}.$$

Then

$$\log y = \frac{1}{x} \log (1 + x),$$

so that

$$\lim_{x \to 0} \log y = \lim_{x \to 0} \left[\frac{1}{x} \log (1 + x) \right] = 1. \qquad \text{Art. 75}$$

Thus

$$\lim_{x \to 0} \log y = 1, \quad \text{so that} \quad \lim_{x \to 0} y = e^1 = e.$$

The desired limit, therefore, has the value e. Compare Art. 24.

EXERCISES

1. Establish each of the following results:

(a) $\lim_{x \to \frac{\pi}{2}} (\pi - 2x) \tan x = 2;$

(b) $\lim_{x \to \frac{\pi}{4}} (1 - \tan x) \sec 2x = 1;$

(c) $\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \frac{1}{3};$

(d) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = 0;$

(e) $\lim_{x \to \infty} (a^{\frac{1}{x}} - 1)x = \log a;$

(f) $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right) = \frac{1}{2};$

(g) $\lim_{x \to 0} (1 + x^2)^{\frac{1}{x}} = 1;$

(h) $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = 1;$

(i) $\lim_{x \to 1} (2 - x)^{\tan \frac{\pi x}{2}} = e^{\frac{2}{\pi}};$

(j) $\lim_{x \to 0} (\log x)^x = 1;$

(k) $\lim_{x \to 0} \left(\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x}\right) = \frac{1}{2};$

(l) $\lim_{x \to \frac{\pi}{2}} \sec mx \cos nx = \frac{n}{m};$

(m) $\lim_{x \to 0} \left(\frac{1}{x}\right)^{\tan x} = 1;$

(n) $\lim_{x \to 1} \tan \frac{\pi x}{2} \log (2 - x) = \frac{2}{\pi}.$

2. Evaluate each of the following limits:

(a) $\lim_{x \to 0} \left(\csc x - \frac{1}{x}\right);$

(b) $\lim_{x \to 0} (\cos x)^{\frac{1}{x^4}};$

(c) $\lim_{x \to 0} (1 + \sin x)^{\cot x};$

(d) $\lim_{x \to \infty} xe^{-x};$

(e) $\lim_{x \to \infty} (1 + x^2)^{\frac{1}{x}};$

(f) $\lim_{x \to \infty} x \sin \frac{a}{x}.$

3. Find $\lim_{x \to 0} (\tan 2x - 2 \tan x)/x^3.$

4. Find $\lim_{x \to \infty} \sqrt{x(x+1)} \ \log_{10} \frac{1+x}{x}.$ *Ans.* $\log_{10} e.$

5. By trying $n = 1, 2, 3, \cdots$, show that 5 is the smallest positive integral value of n for which

$$\lim_{x \to 0} \left\{ \frac{3 \sin x}{2 + \cos x} - x \right\} / x^n$$

is different from zero.

6. Using the general definition of an asymptote as given in Art. 69, it follows that the line $y = a$ will be an asymptote to the curve $y = f(x)$ whenever $\lim_{x \to +\infty} f(x) = a$ or $\lim_{x \to -\infty} f(x) = a$. Draw a figure to indicate the geometrical meaning of this general fact; then apply the fact in particular to establish the following results, drawing an appropriate figure for each:

(a) the line $y = 1$ is an asymptote to the curve $xy = \log x + x$,

(b) the X-axis is an asymptote to any curve of the type $y = x^n e^{-x}$; $n > 0$,

(c) the line $y = \log a$ is an asymptote to the curve $y = x(a^{\frac{1}{x}} - 1)$.

CHAPTER X

INFINITE SERIES

I. Series in General

78. Definitions. A **series** of n terms is an expression of the form

$$u_1 + u_2 + u_3 + \quad + u_n,$$

where the nth term, u_n, is a given function of n.

Illustrations:

$$1^2 + 2^2 + 3^2 + \quad + n^2. \quad \text{Here } u_n = n^2.$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \quad + \frac{1}{2^n}. \quad \text{Here } u_n = \frac{1}{2^n}.$$

In case the number of terms in a series is not stated but is regarded as indefinitely large, the resulting series is known as an **infinite series** and is represented by the symbol

$$u_1 + u_2 + u_3 +$$

Illustrations:

$$1^2 + 2^2 + 3^2 +$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} +$$

$$\frac{1}{2 \cdot 3} - \frac{1}{3 \quad 4} + \frac{1}{4 \cdot 5} -$$

If only a limited number of the early terms of an infinite series be given, it is frequently desirable to ascertain the function of n which represents its nth term. This may often be done by inspection.

Illustrations: In the infinite series

$$\frac{1}{2} + \frac{2^2}{1 + 2^3} + \frac{3^2}{1 + 3^3} + \cdots$$

the nth term is seen to be $\frac{n^2}{1 + n^3}$.

In the infinite series

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

the nth term is seen to be $\frac{(-1)^{n-1}}{(2n - 1)^2}$.

Hereafter the word " series " will be understood to mean " infinite series " unless otherwise stated.

79. Convergence. Divergence. In the series

$$u_1 + u_2 + u_3 + \qquad (1)$$

let us suppose that the sum of the first n terms is represented by S_n: that is, let us place

$$S_n = u_1 + u_2 + u_3 + \quad + u_n.$$

If, then, $\lim\limits_{n \to +\infty} S_n$ has a definite finite value, the series (1) is said to be **convergent.**

For example, consider the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots. \qquad (2)$$

This is recognized at once as a geometric progression whose first term is 1 and whose common ratio is $\frac{1}{2}$. The nth term is $\frac{1}{2^{n-1}}$ and the sum S_n of the first n terms is therefore

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \quad + \frac{1}{2^{n-1}}.$$

But this sum, as shown in elementary studies on the geometric progression,* may be written in the condensed form

$$S_n = \frac{1 - 1 \quad (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1 - (\frac{1}{2})^n}{\frac{1}{2}}.$$

We therefore have

$$\lim_{n \to \infty} S_n = \frac{1 - 0}{\frac{1}{2}} = 2, \qquad (3)$$

thus showing that $\lim\limits_{n \to \infty} S_n$ has the definite finite value 2 and hence establishing the fact that the series (2) is convergent.

If, on the other hand, a series (1) be given for which $\lim\limits_{n \to \infty} S_n$ does *not* have a definite finite value, the series is said to be **divergent.** Such a situation may arise in any one of the three ways as follows:

(1) $\lim\limits_{n \to \infty} S_n = +\infty$ (2) $\lim\limits_{n \to \infty} S_n = -\infty$

(3) $\lim\limits_{n \to \infty} S_n$ has no significance whatever.

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* See, for example, the author's "Brief Course in College Algebra," page 76.

For example, the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \quad + \frac{1}{n} + \cdots \tag{4}$$

can be shown to be divergent as follows:

First group the terms in the following manner:

$$1 + \tfrac{1}{2} + [\tfrac{1}{3} + \tfrac{1}{4}] + [\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}] + [\tfrac{1}{9} + \tfrac{1}{10} + \quad + \tfrac{1}{16}] + \tag{5}$$

The value of each group is seen to be greater than, or equal to, the value of the corresponding group in the series below:

$$\tfrac{1}{2} + \tfrac{1}{2} + [\tfrac{1}{4} + \tfrac{1}{4}] + [\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}] + [\tfrac{1}{16} + \tfrac{1}{16} + \quad + \tfrac{1}{16}] + \tag{6}$$

But each group in (6) has the value $\frac{1}{2}$, so that the sum of these groups can be made as large as one pleases by adding enough of them together. The same property must therefore belong to the series (5). It follows that the sum of the first n terms of the original series (4) can be made as large as one pleases by taking n sufficiently large; that is, $\lim\limits_{n \to \infty} S_n = +\infty$. Therefore the series (4) is divergent.

Again, consider the series

$$1 - 1 + 1 - 1 + 1 - 1 + \quad + (-1)^{n-1} + \cdots.$$

Here the sum of the first two terms is seen to be 0, the sum of the first three is 1, of the first four is again 0, of the first five is again 1, etc. In fact, S_n has the value 0 whenever n is an even number and the value 1 whenever n is an odd number. In this case S_n can evidently approach no definite limit as n increases indefinitely; that is, $\lim\limits_{n \to \infty} S_n$ has no significance. The series is therefore divergent. In general, whenever the expression S_n corresponding to a given series presents an oscillatory character such as here exemplified, thereby failing to approach any definite limit as $n \to +\infty$, the series is said to have **oscillatory divergence.**

80. The sum of a series. If S_n represents the sum of the first n terms of a convergent series, then the limit $\lim\limits_{n \to \infty} S_n$ is called the **sum of the series** and is commonly represented by S; that is, we have by definition

$$S = \lim_{n \to \infty} S_n.$$

As thus defined, the sum of a series is seen to be, strictly speaking, not a sum but the *limit* of a sum.

Thus the sum of the convergent series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} +$$

is 2, since, by (3) of Art. 79, we here have $S = \lim\limits_{n \to \infty} S_n = 2.$

Inasmuch as the limit $\lim\limits_{n\to\infty} S_n$ has no definite value in the case of a divergent series, such series are not regarded as having a sum.* For this reason, convergent series are of central interest in mathematics.

EXERCISES

1. Determine by inspection the nth term of each of the following series:

(a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$

(b) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$

(c) $1 - 4 + 9 - 16 + 25 - \cdots$

(d) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots$

(e) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

(f) $\frac{\sin y}{1^3} - \frac{\sin 3y}{3^3} + \frac{\sin 5y}{5^3} - \cdots$

(g) $1 + x \log a + \frac{x^2 \log^2 a}{1 \cdot 2} + \frac{x^3 \log^3 a}{1 \cdot 2 \cdot 3} + \frac{x^4 \log^4 a}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$

(h) $x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots$

2. Write the first five terms of the series whose nth term is

(a) 3^{n-1}; (c) $\frac{(-1)^n(2n+1)}{n^2+1}$; (e) $\frac{nx^n}{a^n}$;

(b) $\frac{n}{\sqrt{n+1}}$; (d) $\frac{(-1)^{n-1}}{n2^n}$; (f) $\frac{\cos nx}{e^{n^2x}}$.

3. Show that each of the following series is convergent and find its sum:

(a) $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

SOLUTION. Since $\frac{1}{1 \cdot 2} = 1 - \frac{1}{2}$, $\frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}$, $\cdots$, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the sum of the first n terms of the series may be written in the form

$$S_n = \begin{cases} 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ -\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) - \frac{1}{n+1}. \end{cases}$$

Upon noting that the group of terms here appearing in the one parenthesis cancels with that in the other, we now have

$$S_n = 1 - \frac{1}{n+1}.$$

Hence, $\lim\limits_{n\to\infty} S_n = 1$, thus showing that the series in question is convergent and has its sum equal to 1.

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* By changing the definition of the term "sum of a series" it is possible to attribute sums in a consistent manner to many divergent series. Such aspects of series, however, lie beyond the scope of the present book.

(*b*) $\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{(n+2)(n+3)}+\cdots$

HINT. Observe that $\frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4}$, $\frac{1}{4\cdot 5}=\frac{1}{4}-\frac{1}{5}$, etc., and proceed as in the solution of (*a*). It will be found that $S_n=\frac{1}{3}-\frac{1}{n+3}$.

(*c*) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}+\cdots$

HINT.

$$\frac{1}{1\cdot 3}=\frac{1}{2}\left(1-\frac{1}{3}\right),\ \frac{1}{3\cdot 5}=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\ \text{etc.}$$

Thus show that

$$S_n=\frac{1}{2}\left(1-\frac{1}{2n+1}\right).$$

(*d*) $\frac{4}{1\cdot 5}+\frac{4}{5\cdot 9}+\frac{4}{9\cdot 13}+\frac{4}{13\cdot 17}+\cdots+\frac{4}{(4n-3)(4n+1)}+\cdots$

HINT. Show that $S_n=1-\frac{1}{4n+1}$.

(*e*) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)(3n+1)}+\cdots$

HINT. Show that $S_n=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)$.

4. An arithmetic progression is a series of the type form

$$a+(a+d)+(a+2d)+(a+3d)+\cdots+[a+(n-1)d]+\cdots$$

where a and d are given numbers. It is shown in elementary algebra that the sum of the first n terms of this series may be expressed in the form

$$S_n=\frac{n}{2}\{2a+(n-1)d\}.$$

Hence show that every such series is divergent unless both a and d are equal to zero.

5. A geometric progression is a series of the type form

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots$$

where a and r are given numbers. It is shown in elementary algebra that the sum of the first n terms of this series may be expressed in the form

$$S_n=\frac{a-ar^n}{1-r}.$$

Hence show that a geometric progression will converge and have the sum

$$S=\frac{a}{1-r}$$

whenever the value of r lies between -1 and $+1$, but that it will diverge in case r is less than -1 or greater than $+1$.

81. Tests for convergence and divergence. The determination of whether a given series is convergent or divergent cannot be made, as a rule, in such ways as occur in the illustrations in Art. 80 because it is usually difficult, or even impossible, to obtain a simple compact expression representing the sum S_n of the first n terms. In fact, the properties of S_n are commonly quite inaccessible. It therefore becomes desirable to have some means of testing a series for convergence or divergence that can be used immediately from a knowledge of merely the form of the nth term. Many such practical tests have been devised, but we shall confine ourselves in the forthcoming articles to a few of the most important, stating each in the form of a theorem with proof and giving an illustration of its use.

82. A necessary condition for convergence.

Theorem. *The series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{1}$$

can converge only in case $\lim\limits_{n\to\infty} u_n = 0.$

PROOF. To say that the series (1) converges means that the limit $\lim\limits_{n\to\infty} S_n$ exists, having some fixed value S. Under such circumstances the successive values which S_n assumes as n increases indefinitely; that is, the values $S_1, S_2, S_3, S_4, \cdots S_{n-1}, S_n, \cdots$ eventually become and remain different from S, and hence from each other, by as small an amount as one may mention. But, from the definition of S_n, it appears that $S_n - S_{n-1} = u_n$, so that the difference between two successive values of S_n cannot have the property just described concerning it unless $\lim\limits_{n\to\infty} u_n = 0$. The series (1) therefore can converge only in case this condition is satisfied by u_n, as was to be proved.

For example, the series

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \cdots + (-1)^{n-1}\frac{n}{n+1} + \cdots$$

is not convergent; that is, is divergent, since we here have

$$u_n = (-1)^{n-1}\frac{n}{n+1} = \frac{(-1)^{n-1}}{1+\dfrac{1}{n}},$$

and, as n increases indefinitely, this expression does not approach the limit zero. In fact, for large *even* values of n it can be brought as near as we please to the value -1, while for large *odd* values of n it can be brought as near as we please to the value $+1$.

Remark. It is important to observe that the foregoing theorem is useful only in ascertaining that a given series is *not* convergent, for it is not true that a series will always converge if $\lim\limits_{n \to \infty} u_n = 0$.

Thus, in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \quad + \frac{1}{n} +$$

we have $u_n = \frac{1}{n}$ and hence $\lim\limits_{n \to \infty} u_n = 0$, yet this series, as shown in Art. 79, is divergent. Theorem I is therefore to be regarded as stating a *necessary* condition only for convergence.

83. Comparison Tests.

Theorem I. *A series of positive terms*

$$u_1 + u_2 + u_3 + \quad + u_n + \tag{2}$$

will converge whenever it is possible to find a series, already known to be convergent,

$$a_1 + a_2 + a_3 + \quad + a_n + \tag{3}$$

such that the terms u_1, u_2, u_3, *of the given series are respectively less than (or at most equal to) the corresponding terms* a_1, a_2, a_3, *of the latter (test) series.*

PROOF. The proof rests upon the following lemma in the theory of limits, the truth of which is immediately evident geometrically.

Lemma. *If* S_n *is a variable which always increases (or remains unchanged) as* n *increases but which never exceeds some definite number* A, *no matter what value* n *has, then* S_n *approaches a limit* S *as* n *increases indefinitely, and the value of* S *will be less than, or at most equal to,* A.

Thus, in Fig. 98 the successive values S_1, S_2, S_3, S_4, $\cdots$, when laid off to scale on the line OX, arrange themselves from left to right because of the assumption that S_n always increases (or remains unchanged) as n increases.

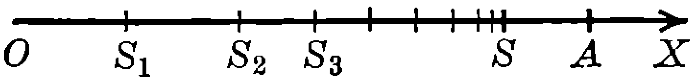

FIG. 98

At the same time, all these values lie to the left of the fixed point A whose distance from O is the given number A. The points corresponding to S_1, S_2, S_3, S_4, must therefore tend to cluster from the left at some fixed point S such as indicated; that is, there must be a fixed value S such that

$$\lim_{n \to \infty} S_n = S.$$

The proof of the theorem is now readily carried out as follows:

Let S_n and T_n represent respectively the sum of the first n terms of the series (2) and (3); also suppose that the sum of the convergent series (3) is T. Then, recalling the hypotheses regarding the nature of the terms u_n and a_n, we have

$$S_n \leqq T_n \quad \text{and} \quad T_n < T.$$

Therefore,

$$S_n < T.$$

The variable S_n, which increases as n increases, thus remains always less than the fixed number T, from which we conclude by the foregoing lemma that $\lim_{n \to \infty} S_n$ exists; that is, the series (2) is convergent, as was to be proved.

Remarks. Theorem I furnishes what may be regarded as a *sufficient* condition for convergence; that is, if the condition as stated is fulfilled, this fact suffices to assure one that convergence is present. It is not true, however, that the condition *must* be satisfied in order that convergence be present; that is, the condition as stated is not a *necessary* one. Compare Art. 82.

It is also to be observed that a series (2) will converge in case the conditions stated in Theorem I are satisfied after any particular term, as u_p, rather than from the beginning. The convergence or divergence of a series is, in fact, evidently not altered by disregarding any given number of terms at the beginning.

Corresponding to Theorem I relating to convergence, we have also the following relating to divergence:

Theorem II. *A series of positive terms*

$$u_1 + u_2 + u_3 + \quad + u_n + \cdots \tag{2}$$

will diverge whenever it is possible to find a series, already known to be divergent,

$$a_1 + a_2 + a_3 + \quad + a_n + \tag{3}$$

such that the terms u_1, u_2, u_3, *of the given series are respectively greater than (or at least equal to) the corresponding terms a_1, a_2, a_3,* *of the latter (test) series.*

PROOF. Under the given conditions the sum S_n of the first n terms of (2) is greater than, or at least equal to, the sum of the first n terms of (3). But the latter sum increases indefinitely as n increases. Hence, the same must be true of S_n. Hence (2) is divergent.

For example, the series of positive terms

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\quad+\frac{1}{\sqrt{n}}+$$

is divergent since each of its terms is greater than, or at least equal to, the corresponding term of the divergent series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\quad+\frac{1}{n}+\cdots.$$

On the other hand, the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\quad+\frac{1}{n^2}+$$

is convergent, as follows by comparison with (2) of Art. 79.

The tests for convergence and divergence which are furnished by Theorems I and II are known as **comparison tests** inasmuch as they involve a comparison of the terms of the given series with those of another series whose convergence properties are regarded as known.

84. Ratio test. There are many so-called "ratio tests" for determining whether a given series is convergent or divergent. The simplest and most frequently used one may be stated as follows:

Theorem. *Given the series to be tested*

$$u_1+u_2+u_3+\quad+u_n+ \tag{1}$$

Form the ratio $\frac{u_{n+1}}{u_n}$ *of the* $(n+1)$*st term to the nth term and consider its absolute value; namely,* $\left|\frac{u_{n+1}}{u_n}\right|$.* Then

(*a*) *if* $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$ *is less than* 1, *the series converges;*

(*b*) *if* $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$ *is greater than* 1, *or if* $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=+\infty$, *the series diverges;*

(*c*) *if* $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=1$, *no conclusion can be drawn; that is, the test fails.*

* The absolute, or numerical, value of any real number is defined as its value when its sign, whether positive or negative, is taken as positive. Thus $|+2|=+2$, $|-2|=+2$, $|-5|=+5$, $|+5|=+5$, etc. See also Art. 47.

PROOF. In order to prove part (*a*) let us suppose at first that the given series (1) contains positive terms only, thus making $\left|\frac{u_{n+1}}{u_n}\right| = \frac{u_{n+1}}{u_n}$. We then have

$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = L,$$

where L is some number lying between zero and 1. Now let r be any fixed number whose value lies between L and 1. Then, inasmuch as $\frac{u_{n+1}}{u_n}$ approaches L as a limit as $n \to \infty$, it follows from the definition of a limit (Art. 4) that the difference between $\frac{u_{n+1}}{u_n}$ and L can be made as small as we please by taking n sufficiently large. In particular a number m can be found such that for all values of n greater than m we shall have

$$\frac{u_{n+1}}{u_n} < r.^*$$

Letting n take in succession the values m, $m + 1$, $m + 2$, etc., we thus have

$$u_{m+1} < u_m r,$$
$$u_{m+2} < u_{m+1} r < u_m r^2,$$
$$u_{m+3} < u_{m+2} r < u_m r^3,$$

It follows that the terms of the series (1) following the term u_m are less respectively than the corresponding terms of the series

$$u_m r + u_m r^2 + u_m r^3 + \qquad (2)$$

But this series, being a geometric series in which the common ratio r is less than 1, is convergent (see Ex. 5, page 161). Hence, noting that under the present hypotheses both series (1) and (2) are series of positive terms, we may now apply the theorem of Art. 83 and show that (1) is convergent.

---

* The student is reminded that the symbols < and > are used respectively to denote "less than" and "greater than."

The proof of part (*a*) in case the series (1) does not have all its terms positive can best be given after Theorem II of the next article has been stated, hence it will be delayed until that point.

In case (*b*) it is possible, reasoning as in case (*a*), to find a positive number r greater than 1, such that for all values of n greater than some fixed value m we may write, whatever the signs of the terms of (1),

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|u_{n+1}|}{|u_n|} > r > 1,$$

so that

$$|u_{n+1}| > |u_n|; \quad n = m, m+1, m+2,$$

Hence, all the terms of (1) are eventually greater numerically than the constant $|u_m|$. Therefore they do not approach zero as a limit as n increases indefinitely. It follows by the theorem of Art. 82 that (1) is necessarily divergent.

The fact that no inference can be drawn when the hypotheses of part (*c*) are present follows from the fact, as will now be shown, that both convergent and divergent series (1) may be found such that $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$. Thus, for the divergent series of positive terms

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \quad + \frac{1}{n} +$$

we have

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}; \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{1+0} = 1,$$

while for the convergent series (see Art. 83).

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \quad + \frac{1}{n^2} +$$

we have likewise

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{u_{n+1}}{u_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\frac{1}{n}\right)^2};$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{(1+0)^2} = 1.$$

Example 1. Apply the ratio test to the series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{(n-1)!}+\cdots.$$

SOLUTION. We have $u_n = \dfrac{1}{(n-1)!}$ and therefore $u_{n+1} = \dfrac{1}{n!}\cdot$ Hence

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{u_{n+1}}{u_n} = \frac{\dfrac{1}{n!}}{\dfrac{1}{(n-1)!}} = \frac{(n-1)!}{n!} = \frac{1\cdot 2\cdot 3\cdots(n-1)}{1\cdot 2\cdot 3\cdots(n-1)n} = \frac{1}{n}.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0.$$

Hence, by part (*a*) of the theorem, the series is convergent.

Example 2. Apply the ratio test to the series

$$\frac{1}{3}-\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}-\frac{1\cdot 3\cdot 5\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{3^n\cdot n!}+\cdots.$$

SOLUTION. We have

$$u_n = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{3^n\cdot n!}$$

and hence $$u_{n+1} = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{3^{n+1}\cdot(n+1)!}.$$

Hence we find, upon observing extensive cancellation of factors, that

$$\frac{u_{n+1}}{u_n} = -\frac{2n+1}{3(n+1)}.$$

It follows that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{2n+1}{3(n+1)} = \lim_{n\to\infty}\frac{2+\dfrac{1}{n}}{3\left(1+\dfrac{1}{n}\right)} = \frac{2}{3}.$$

Hence, by part (*a*) of the theorem, the series is convergent.

---

*The student will recall that the symbol $n!$ (read "factorial n") represents the product of the positive integers from 1 to n inclusive. Thus, $1! = 1$; $2! = 1\cdot 2$; $3! = 1\cdot 2\cdot 3$; $4! = 1\cdot 2\cdot 3\cdot 4$; etc. In general, we have $n! = n\cdot(n-1)!$ or $(n-1)! = \dfrac{n!}{n}$, from which it follows in particular by placing $n = 1$ that $0! = \dfrac{1!}{1} = 1$.

Example 3. Apply the ratio test to the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots.$$

SOLUTION. We have $u_n = \frac{1}{n(n+1)}$, hence $u_{n+1} = \frac{1}{(n+1)(n+2)}$.

Therefore,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{(n+1)(n+2)}{n(n+1)} = \lim_{n\to\infty} 1 + \frac{2}{n} = 1.$$

By part (*c*) of the theorem, the test fails. However, the series is convergent, as may be shown by other means. See Ex. 3 (*a*), page 166.

85. Alternating series. Absolutely convergent series. The name **alternating series** is given to any series whose terms are alternately positive and negative; that is, to any series of the type form

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n + \cdots \tag{1}$$

where the quantities $u_1, u_2, u_3, u_4, \cdots$ are all positive. Such series are frequently met with in practice and for them the following special theorem, proof of which will be omitted for brevity, holds true:

Theorem I. *An alternating series* (1) *will converge whenever* (*a*) *the successive quantities* $u_1, u_2, u_3, u_4, \cdots$ *continually diminish in value, and* (*b*) *the condition* $\lim_{n\to\infty} u_n = 0$ *is satisfied.*

Thus, the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots \tag{2}$$

is at once known to be convergent since the positive values

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n},$$

continually diminish as one proceeds to the right and we have moreover $\lim_{n\to\infty} \frac{1}{n} = 0$.

A series which contains both positive and negative terms is said to be **absolutely convergent** in case the series formed from it by making all its terms positive is convergent. In this connection, the following theorem, proof of which is again omitted for brevity, is frequently useful.

Theorem II. *In order to show that a given series is convergent it suffices to show that it is absolutely convergent.*

Thus, the series

$$1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} -$$

is convergent since it is absolutely convergent. In fact, the series obtained from this series by giving the positive sign to each of its terms is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} +$$

and this is readily shown to be convergent.

An immediate corollary of this theorem is the fact that part (*a*) of the theorem of Art. 84 holds true not only when the terms of the series are all positive (see the proof as given in that article), but holds true equally well no matter what the signs of the terms may be. The steps of such a proof being easily supplied, are left to the student.

EXERCISES

1. Apply the theorem of Art. 82 to show that each of the following series must be divergent:

(*a*) $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} +$

(*b*) $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} +$

(*c*) $\log 2 - \log 3 + \log 4 - \log 5 + \cdots;$

(*d*) $1 + x + x^2 + x^3 +$ *where* $|x| \geqq 1;$

(*e*) $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \cdots$, *where* $|x| > 1;$

(*f*) $\sin x + \sin 2x + \sin 3x +$ *where* x *has any value except* $0, \pm\pi, \pm 2\pi, \pm 3\pi$, etc.

2. Apply the comparison tests (Art. 83) to determine which of the following series are convergent and which are divergent:

(*a*) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} +$ *Ans.* Conv.

(*b*) $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$, *where* $p > 2;$

(*c*) $\frac{1}{2^3} + \frac{2}{3^3} + \frac{3}{4^3} + \frac{4}{5^3} +$

HINT. Compare with the series which results when each numerator is made greater by 1.

(*d*) $\frac{3}{2} + \frac{4}{2\cdot 3} + \frac{5}{3\cdot 4} + \frac{6}{4\cdot 5} + \cdots.$

3. Apply the ratio test (Art. 84) to determine which of the following series are convergent and which are divergent:

(*a*) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \quad + \frac{n}{2^{n-1}} +$ *Ans.* Conv.

(*b*) $1 - \frac{2}{3} + \frac{3}{3^2} - \frac{4}{3^3} + \frac{5}{3^4} - \cdots + (-1)^{n-1}\frac{n}{3^{n-1}} + \cdots;$

(*c*) $\frac{2!}{10} + \frac{3!}{10^2} + \frac{4!}{10^3} + \frac{5!}{10^4} + \quad + \frac{(n+1)!}{10^n} +$

(*d*) $3 - \frac{3^2}{1!} + \frac{3^3}{2!} - \frac{3^4}{3!} + \frac{3^5}{4!} - \cdots + (-1)^{n-1}\frac{3^n}{(n-1)!} +$

4. Investigate the convergence of each of the following series by means of Theorem I or II (or both) of Art. 85:

(*a*) $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} +$

(*b*) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} +$

(*c*) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots;$

(*d*) $\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} +$

5. Investigate the convergence (or divergence) of each of the following series, using any one of the foregoing methods:

(*a*) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} +$

(*b*) $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} +$

(*c*) $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} +$

(*d*) $\frac{1}{3 \cdot 2^2} + \frac{2}{4 \cdot 3^2} + \frac{3}{5 \cdot 4^2} +$

(*e*) $\frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \frac{5^2}{4!} +$

(*f*) $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \quad + \frac{1}{n^2+1} +$

6. Investigate the convergence of the series

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \quad + \frac{n^2-1}{n^2+1}x^n +$$

when $|x| < 1$. Consider likewise the case in which $|x| > 1$. What can be said in case $|x| = 1$?

86. Evaluation of the sum of a series. In Art. 80 the sum S of a series was defined as the value of the limit $\lim_{n\to\infty} S_n$, where S_n represents the sum of the first n terms. Moreover, illustrations were given of series for which the sum as thus defined could be readily evaluated. In accordance with this definition every series for which $\lim_{n\to\infty} S_n$ has a definite meaning, that is, every convergent series, has a sum, but it is now to be noted that, unlike the special illustrations just

cited, the value of the sum is commonly difficult or even impossible to determine exactly. It is often possible, however, to obtain the sum of a series correct to any prescribed degree of approximation (for example, with a possible error of less than one ten-thousandth) by adding together a limited number of the early terms. The process is illustrated in the following example.

Example 1. Evaluate the sum of the following convergent series correct to four places of decimals.

$$\frac{1}{1 \cdot 7} + \frac{1}{2 \cdot 7^2} + \frac{1}{3 \cdot 7^3} + \frac{1}{4 \cdot 7^4} + \frac{1}{5 \cdot 7^5} + \frac{1}{6 \cdot 7^6} + \cdots + \frac{1}{n \cdot 7^n} + \cdots \qquad (1)$$

SOLUTION. Evaluating each of the first four terms correct to seven places of decimals, we find

$$\frac{1}{1 \cdot 7} = 0.1428571, \qquad \frac{1}{2 \cdot 7^2} = 0.0102040,$$
$$\frac{1}{3 \cdot 7^3} = 0.0009718, \qquad \frac{1}{4 \cdot 7^4} = 0.0001041,$$

and the sum of these we find to be 0.154137. Furthermore, the terms of (1) beyond the fourth are seen to diminish steadily in value as one proceeds to the right in the series, and, as the largest of them is less than $\frac{1}{4 \cdot 7^4} = 0.0001041$, it is reasonable to expect that the combined effect of these additional terms upon the sum of (1) lies beyond the fourth decimal place. Hence, we may assume tentatively that the sum of (1), correct to four decimal places, is 0.1541. Actually to confirm this conclusion, it now suffices to reason as follows:

The terms of (1) beyond the fourth constitute the series

$$\frac{1}{5 \cdot 7^5} + \frac{1}{6 \cdot 7^6} + \frac{1}{7 \cdot 7^7} + \cdots$$

and these terms are respectively less than (or at most equal to) those of the geometric progression

$$\frac{1}{5 \cdot 7^5} + \frac{1}{5 \cdot 7^6} + \frac{1}{5 \cdot 7^7} + \cdots = \frac{1}{5 \cdot 7^5}\left[1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots\right],$$

the sum of which, by Ex. 5, page 161, is

$$\frac{1}{5 \cdot 7^5}\left[\frac{1}{1 - \frac{1}{7}}\right] = \frac{1}{5 \cdot 7^5} \cdot \frac{7}{6} = \frac{1}{30 \cdot 7^4} = \frac{1}{72{,}030} = 0.000013^+.$$

The addition of 0.000013^+ to the value previously mentioned, namely, to 0.154137, evidently affects it only beyond the fourth decimal place, as was to be shown.

In summary, then, the sum of the series (1) correct to four places of decimals is 0.1541.

While no general rule can be stated for evaluating the sum of a given series correct to a prescribed number of decimal places, a careful examination of the foregoing example will make clear the following statements.

Having given any convergent series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{2}$$

and desiring to evaluate its sum correct to k places of decimals, one may begin evaluating decimally to several places beyond the kth a number of the early terms $u_1, u_2, u_3, \cdots$ of the series. Upon adding a sufficiently large number, m say, of these, a tentative value of S correct to k places of decimals results. To confirm this tentative result, the series remaining from (2) after the removal of its first m terms (so-called *remainder series*) must be examined and, by some method more or less ingenious, its sum must be shown to be so small as to have no effect upon the tentative sum above mentioned when added to it except beyond the kth decimal place. This procedure will always succeed in giving S to the desired degree of approximation, provided a sufficiently large value of m is employed.

A series for which it suffices to use a relatively small number m of the early terms in order to obtain the sum of the series correct to a given high degree of approximation is said to **converge rapidly**, while a series is said to **converge slowly** in case a relatively large value of m is similarly required.

Thus, the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

converges rapidly as compared to the series

$$1 + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$$

and converges still more rapidly as compared to the series

$$1 + \frac{1}{1^{\frac{3}{2}}} + \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \frac{1}{4^{\frac{3}{2}}} + \cdots$$

The procedure outlined above may be greatly simplified in case the given series (2) is an alternating series whose terms satisfy the conditions mentioned in Theorem I of Art. 85. In fact, the following general theorem may then be employed to advantage, the proof of the theorem being here omitted, however, for the sake of brevity:

Theorem. *If S represents the sum of the convergent alternating series*

$$u_1 - u_2 + u_3 - u_4 + u_5 - \cdots + (-1)^{n-1}u_n + (-1)^n u_{n+1} + \cdots$$

wherein it is assumed (as in Theorem I of Art. 85) that (a) the successive quantities u_1, u_2, u_3, $\cdots$ continually diminish in value and (b) that $\lim\limits_{n\to\infty} u_n = 0$, then the sum of the first n terms of the series, whatever value be chosen for n, gives the value of S correct within a possible error not exceeding the numerical value of the first term neglected; that is, an error not greater than u_{n+1}.

Thus, having given the convergent alternating series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots$$

we may assert that the sum S differs from

$$1 - \tfrac{1}{2} = \tfrac{1}{2}$$

by less than $\frac{1}{3}$; similarly S differs from

$$1 - \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}$$

by less than $\frac{1}{4}$; again, S differs from

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{7}{12}$$

by less than $\frac{1}{5}$; etc. Evidently, by summing a sufficiently large number of the early terms of the series the value of S may thus be obtained to any prescribed degree of accuracy.

The manner in which the foregoing theorem is used in practice is illustrated in the following example:

Example 2. Evaluate the sum of the following alternating series correct to four decimal places (compare with series (1)).

$$\frac{1}{1 \cdot 7} - \frac{1}{2 \cdot 7^2} + \frac{1}{3 \cdot 7^3} - \frac{1}{4 \cdot 7^4} + \frac{1}{5 \cdot 7^5} - \cdots \qquad (3)$$

SOLUTION. Since $5 \cdot 7^5 = 84{,}035$, the term $\dfrac{1}{5 \cdot 7^5}$ is equal to $\dfrac{1}{84035}$, which is equal decimally to 0.000011^+. Hence (observing that the terms of (3) satisfy the conditions of the theorem) we may assert that the error committed by taking for the sum S of (3) the sum merely of its first four terms is less than 0.000012, which value lies beyond the fourth decimal place.

Evaluating decimally the sum of the first four terms of the series, using for this purpose the values $\dfrac{1}{1 \cdot 7} = 0.1428571$, $\dfrac{1}{2 \cdot 7^2} = 0.0102040$, etc., indicated in the solution of Example 1, we thus find that the desired sum of the series, correct to four decimal places, is 0.1335.

EXERCISES

Evaluate, correct to the decimal place indicated, the sum S of each of the following convergent series:

1. $\frac{1}{1\ 9} + \frac{1}{2\ 9^2} + \frac{1}{3\ 9^3} + \frac{1}{4\ 9^4} +$ *(four places)*

2. $\frac{1}{1\ 2} + \frac{1}{2\ 3}\ \frac{1}{10} + \frac{1}{3\ 4}\ \frac{1}{10^2} + \frac{1}{4\ 5}\ \frac{1}{10^3} +$ *(four places)*

3. $1 + \frac{1}{3} + \frac{1}{3^2\ 2!} + \frac{1}{3^3\ 3!} + \frac{1}{3^4\ 4!} +$ *(three places)*

4. $\frac{1}{1\ 2} - \frac{1}{2\ 3}\ \frac{1}{10} + \frac{1}{3\ 4}\ \frac{1}{10^2} - \frac{1}{4\ 5}\ \frac{1}{10^3} + \cdots$ *(four places)*

5. $1 - \frac{1}{3} + \frac{1}{3^2\ 2!} - \frac{1}{3^3\ 3!} + \frac{1}{3^4\ 4!} -$ *(three places)*

6. $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} +$ *(four places)*

7. $1 + \frac{1}{2^2\ 2!} + \frac{1}{3^3\ 3!} + \frac{1}{4^4\ 4!} +$ *(three places)*

II. POWER SERIES

87. Definition. Of especial importance in the calculus and all branches of higher mathematics is the special class of series known as **power series.** Such series, by definition, comprise all those of the type-form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \qquad + a_nx^n +$$

where the quantities (coefficients) a_0, a_1, a_2, $\quad a_n$, are constants but where x is regarded as a variable. The appropriateness of the name "power series" is seen to lie in the manner in which the various ascending powers of x are contained in the succeeding terms of the series.

From the foregoing definition it is evident that whether or not a given power series is convergent or divergent will depend, in general, upon the value assigned to the variable x. In fact, such a series will usually converge for some values of x and diverge for others.

Thus, the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \cdots$$

can be shown to be convergent for every value of x lying between -1 and $+1$ but divergent for every value of x lying outside this interval; that is, divergence is present for every value which is either greater than $+1$ or less than -1. These are special results which follow directly from a general theorem concerning power series which will now be established.

88. The convergence of power series. The following general theorem may now be established relative to any given power series:

Theorem. *The power series*

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \quad + a_nx^n + \tag{1}$$

being given, let L be the value of the limit $\lim\limits_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right|$. *Then, the series will*

(*a*) *converge for every value of x such that* $|x| < L$;

(*b*) *diverge for every value of x such that* $|x| > L$.

PROOF. We first observe that the nth term, u_n, of (1) is $u_n = a_{n-1}x^{n-1}$. Hence $u_{n+1} = a_nx^n$ and consequently we may write

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{a_nx^n}{a_{n-1}x^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| \cdot |x| = \frac{|x|}{\lim\limits_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right|}.$$

Therefore, by the ratio test of Art. 84 the series (1), when considered for any value of x, will (*a*) converge in case the last expression here appearing is < 1; (*b*) diverge in case the same expression is > 1. But these results are seen to be equivalent to statements (*a*) and (*b*).

Remark. The geometrical meaning of the theorem is as follows:

Let XX' be a straight line of indefinite extent which has been laid off to scale to the right and the left of some point O regarded as origin; also let x represent the single coördinate (abscissa) of any point P upon this line. Now, lay off to the right of O a length of L units, where L has the value indicated in the theorem, and do the same to the left of O. There are thus determined two fixed points, $x = L$ and $x = -L$ and the interval between them, which is designated as $(-L, L)$. For any value of x pertaining to the *interior* of this interval we evidently have $|x| < L$ and hence, by the theorem, the series (1) will converge for this value of x. On the other hand, it appears likewise that for any value of x lying *outside* this interval we have $|x| > L$ and hence the series (1) will diverge. For these reasons, the interval $(-L, L)$ pertaining to any given series (1) is called its **interval of convergence.**

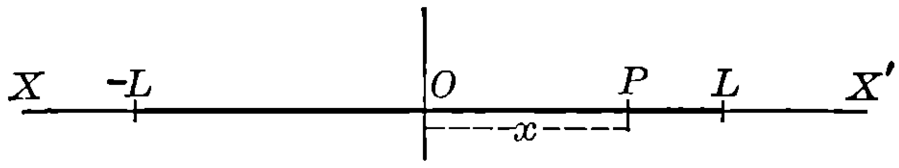

FIG. 99

It is to be observed that the foregoing theorem gives no information as to convergence or divergence when $|x| = L$; that is, when $x = -L$ or $x = L$, these values corresponding to the end-points of the interval

of convergence. For each of these two exceptional values of x the series (1) must be examined independently by means of the theorems of Arts. 81–85 or by other means.

Finally, the limit indicated by L may be $+\infty$ instead of a definite value. In this case, the corresponding series (1) converges for all values of x. Otherwise stated, the interval of convergence is infinite.

Example 1. Determine the interval of convergence of the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \quad + \frac{x^n}{n+1} + \tag{2}$$

SOLUTION. In the notation of the theorem we here have $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{4}$, $\cdots$, $a_n = \frac{1}{n+1}\cdot$ Hence, $a_{n-1} = \frac{1}{n}$ and therefore

$$L = \lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

Hence, applying the theorem, the series (2) converges for every value of x such that $|x| < 1$ and it diverges for every value such that $|x| > 1$; that is, the interval of convergence is $(-1, 1)$ as represented in Fig. 100. In the present instance one may also determine without difficulty the convergence properties of the series at the end-points $x = -1$ and $x = 1$. Thus, when $x = -1$ the series becomes identical with (2) of Art. 85 and is therefore convergent. Again, when $x = 1$ the series becomes identical with (4) of Art. 79 and is therefore divergent.

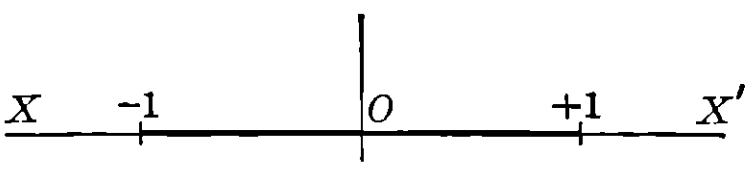

FIG. 100

In summary, the series (2) is convergent whenever $-1 \leqq x < 1$ and divergent for all other values of x. Compare this final result with the statement at the end of Art. 87.

Example 2. Determine the interval of convergence of the power series

$$1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \quad + \frac{x^n}{(n+1)!} + \tag{3}$$

SOLUTION. Here we have $a_n = \frac{1}{(n+1)!}$ and hence $a_{n-1} = \frac{1}{n!}\cdot$ Therefore,

$$L = \lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty} |n+1| = +\infty.$$

The interval of convergence of (3) is therefore infinite; that is, the series converges for all values (positive or negative) of x.

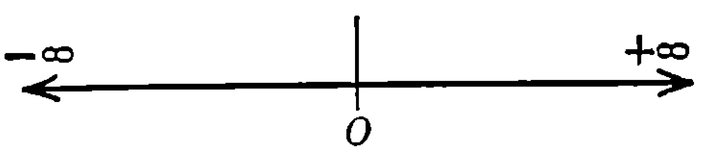

FIG. 101

89. Maclaurin's Series. The value of the sum S of a power series depends, as indicated in Art. 87, upon the value assigned to the variable x appearing in the series, hence such a sum is, in reality, a function of x. From this point of view, the following problem now suggests itself: How shall the power series be formed whose sum S shall be a *given* function of x, say $f(x)$; that is, how shall the constants $a_0, a_1, a_2, \cdots, a_n,$ be determined so that the following equation may exist, it being understood that x is within the interval of convergence:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \quad + a_nx^n + \qquad (1)$$

This is a problem of prime importance, as will appear presently, but it presents certain preliminary difficulties, since not every function $f(x)$ can be thus "developed" or "expanded" into a power series, as will also appear presently. It is not difficult, however, to show what the value of each of the constants (coefficients) $a_0, a_1, a_2, a_3, \quad a_n$ must be in case such an equation as (1) *is* possible. We may, in fact, now establish the following theorem:

Theorem. *If the function $f(x)$ is developable into a power series in x, the series must necessarily have the form*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots, \qquad (2)$$

where $f(0), f'(0), f''(0), f'''(0), \cdots, f^{(n)}(0),$ represent respectively the values of $f(x)$ and its successive derivatives when $x = 0$.

PROOF. Since, by hypothesis, a development of $f(x)$ in the form (1) exists, we have but to place $x = 0$ in both members of (1) to obtain $f(0) = a_0$; that is, a_0 must have the value $f(0)$ as indicated in (2). Again, differentiating (1) member by member we obtain

$$f'(x) = a_1 + 2\,a_2x + 3\,a_3x^2 + \quad + na_nx^{n-1} + \qquad (3)$$

and, placing $x = 0$ in this result, we obtain $f'(0) = a_1$. Hence $a_1 = f'(0)$, or $a_1 = \dfrac{f'(0)}{1!}$, as indicated in (2). Similarly, from (3)

$$f''(x) = 1 \cdot 2\,a_2 + 2 \cdot 3\,a_3x + 3 \cdot 4\,a_4x^2 + \quad + n(n-1)a_nx^{n-2} +$$

and hence $f''(0) = 1 \cdot 2\,a_2 = 2!a_2$, so that $a_2 = \dfrac{f''(0)}{2!}$, as indicated in (2). Evidently the same process may be continued indefinitely, leading to the general result $a_n = \dfrac{f^{(n)}(0)}{n!}$ with which the proof is complete.

Remarks. The series (2) corresponding to a given function $f(x)$ is known as the **Maclaurin's series** * for this function. It may evidently be always written out formally; that is, without regard to convergence or other properties, directly from a knowledge of $f(x)$ itself. Moreover, it is shown in more advanced texts that for all ordinary functions $f(x)$ (such functions, for example, as we are concerned with in this text) the corresponding Maclaurin series converges to the value $f(x)$, as desired, for every value of x within the interval of convergence. We may, therefore, safely use formula (2) to obtain actual power series developments, or expansions, of given functions.

Example. Develop the function e^{-x} in Maclaurin series.

SOLUTION. Here we have

$$\begin{aligned}
f(x) &= e^{-x}, &\quad \text{hence}\quad & f(0) = e^0 = 1,\\
f'(x) &= -e^{-x}, &\quad \text{hence}\quad & f'(0) = -e^0 = -1,\\
f''(x) &= e^{-x}, &\quad \text{hence}\quad & f''(0) = 1,\\
f'''(x) &= -e^{-x}, &\quad \text{hence}\quad & f'''(0) = -1,\\
f^{(n)}(x) &= (-1)^n e^{-x}, &\quad \text{hence}\quad & f^{(n)}(0) = (-1)^n.
\end{aligned}$$

Substituting in (2) the values thus found for $f(0), f'(0), f''(0)$, etc., we obtain as the desired development

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n}{n!}x^n + \cdots$$

This series is readily shown by the theorem of Art. 88 to be convergent for every value (positive or negative) of x; hence, in accordance with the foregoing general remarks, it may be used to compute to any desired degree of accuracy the values of e^{-x}. For example,

$$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \tag{4}$$

Now, $\frac{1}{2!} = \frac{1}{2} = 0.5$, $\frac{1}{3!} = \frac{1}{6} = 0.166667$, $\frac{1}{4!} = \frac{1}{24} = 0.041667$, $\frac{1}{5!} = \frac{1}{120}$ $= 0.008333$, $\frac{1}{6!} = \frac{1}{720} = 0.001389$, $\frac{1}{7!} = \frac{1}{5040} = 0.000198$, $\frac{1}{8!} = \frac{1}{40320} =$ 0.000025, $\frac{1}{9!} = 0.000003$. Substituting these values in (4) and making use of the theorem of Art. 86, it appears that the value of $1/e$ correct to five places of decimals is 0.36788. Hence,

$$e = 1 \div 0.36788 = 2.71828^{+}.$$

Thus we have obtained the value for e which was mentioned in Art. 24 but which we were not able to obtain readily at that point.

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* *Colin Maclaurin* (1698–1746) was a professor at the University of Edinburgh.

EXERCISES

1. Find, by the theorem of Art. 88, the interval of convergence of each of the following power series, illustrating each by a diagram:

(*a*) $1 + x + x^2 + x^3 + \quad + x^n +$ *Ans.* $-1 < x < 1$.

(*b*) $1 - \frac{x}{3} + \frac{x^2}{3^2} - \frac{x^3}{3^3} + \quad + \frac{(-1)^n}{3^n} x^n +$ *Ans.* $-3 < x < 3$.

(*c*) $1 + 2x + 3x^2 + \quad + (n+1)x^n +$

(*d*) $1 + \frac{2x}{2} + \frac{3x^2}{4} + \frac{4x^3}{8} + \quad + \frac{n+1}{2^n} x^n +$

(*e*) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \quad + \frac{(-1)^{n-1}}{n} x^n +$

(*f*) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \quad + \frac{x^n}{n^2+1} +$

(*g*) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \quad + \frac{x^n}{n!} +$

2. Obtain each of the following Maclaurin series developments:

(*a*) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \quad + \frac{x^n}{n!} +$

(*b*) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \quad + \frac{(-1)^{n-1}}{n} x^n +$

(*c*) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \quad + \frac{(-1)^n}{(2n+1)!} x^{2n+1} +$

(*d*) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \quad + \frac{(-1)^n}{(2n)!} x^{2n} +$

(*e*) $a^x = 1 + x \log a + \frac{(\log a)^2}{2!} x^2 + \frac{(\log a)^3}{3!} x^3 +$

(*f*) $\text{arc} \sin x = x + \frac{1}{2\ 3} x^3 + \frac{1\ 3}{2\ 4\ 5} x^5 + \frac{1\ 3\ 5}{2\ 4\ 6\ 7} x^7 +$

(*g*) $\text{arc} \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$

3. By means of the series developments in Ex. 2 obtain the following numerical results:

(*a*) $e = 2.7182$

HINT. Place $x = 1$ in Ex. 2 (*a*). Make use of the values of $\frac{1}{2!}, \frac{1}{3!}$, etc. given in the Example of Art. 89 and compare the method there used for computing e with the one now suggested.

(*b*) $\log_e 1.06 = 0.058$

(*c*) $\sin 10° = 0.1736$

HINT. $10° = 10 \times \frac{\pi}{180} = \frac{\pi}{18} = 0.17453^+$, hence use the series in Ex. 2 (*c*) with $x = 0.17453$.

(*d*) $\cos 10° = 0.9848$

(*e*) $\text{arc} \tan \frac{1}{5} = 0.1973$ radians $= 11° 19'$.

4. By means of the theorem in Art. 89 obtain the binomial expansion

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 +$$
$$+ \qquad + \frac{m(m-1)(m-2) \qquad (m-n+1)}{n!}x^n +$$

Show also that the interval of convergence of this series is $-1 < x < 1$.

5. Show (by trial) that the function $\log x$ cannot be expanded in Maclaurin Test similarly the functions $\frac{1}{x}$, $\cot x$, $x \cot x$.

6. By means of the result in Ex. 2 (*f*) show that $\frac{\pi}{6}$ may be expressed in series form as follows:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\ \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\frac{3}{4}\ \frac{1}{5}\left(\frac{1}{2}\right)^5 +$$

Confirm by this result the well-known fact that $\pi = 3.1415^+$.

7. Obtain each of the following Maclaurin developments, and establish for each the indicated interval of convergence:

(*a*) $\log\frac{1+x}{1-x} = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \qquad\right)$; $\qquad (-1 < x < 1)$;

(*b*) $\sin^2 x = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \qquad + (-1)^{n-1}\frac{2^{2n-1}}{(2n)!}x^{2n} +$

$(-\infty < x < \infty)$;

HINT. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

(*c*) $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \qquad + (-1)^n\frac{2^{2n-1}}{(2n)!}x^{2n} + \cdots$; $(-\infty < x < \infty)$;

(*d*) $e^x \sin x = x + x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 - \frac{8}{6!}x^6 - \qquad + \frac{(\sqrt{2})^n \sin\frac{n\pi}{4}}{n!}x^n +$

$(-\infty < x < \infty)$.

8. By use of the theorem of Art. 84, show that the interval of convergence for each of the series in Ex. 2 (*a*), (*c*), (*d*), (*e*) is infinite, while for each of the series of Ex. 2 (*b*), (*f*), (*g*) the interval is $-1 < x < 1$.

90. Taylor's series. Instead of developing a given function $f(x)$ in a series which proceeds according to ascending powers of x, as in Maclaurin's series, it is often convenient to develop it in a series proceeding according to ascending powers of $x - a$, where a is some given number; that is, to develop $f(x)$ in the form

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 +$$
$$+ b_n(x-a)^n + \qquad (1)$$

where the quantities (coefficients) b_0, b_1, b_2, $\quad b_n$, are constants. In this connection we have the following theorem, analogous to that of Art. 89, for determining the specific form of such a series:

Theorem. *If the function $f(x)$ is developable into a power series in $x - a$, the series must necessarily have the form*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots, \qquad (2)$$

where $f(a), f'(a), f''(a), f'''(a), \quad f^{(n)}(a),$ represent respectively the values of $f(x)$ and its successive derivatives when evaluated at $x = a$.

PROOF. The proof, which closely resembles that of the theorem in Art. 89, may be carried out as follows:

Since, by hypothesis, a development of $f(x)$ in the form (1) exists, we have but to place $x = a$ in both members of (1) to obtain $f(a) = b_0$; that is, $b_0 = f(a)$, as indicated in (2). Again, differentiating (1) member by member, we obtain

$$f'(x) = b_1 + 2\,b_2(x-a) + 3\,b_3(x-a)^2 +$$
$$+ nb_n(x-a)^{n-1} + \qquad (3)$$

and, placing $x = a$ in this result, we obtain $f'(a) = b_1$. Hence, $b_1 = f'(a)$, which may be written $b_1 = \frac{f'(a)}{1!}$, as indicated in (2). Similarly, differentiating (3), we obtain

$$f''(x) = 1 \quad 2\,b_2 + 2 \quad 3\,b_3(x-a) + 3 \quad 4\,b_4(x-a)^2 +$$
$$+ n(n-1)b_n(x-a)^{n-2} +$$

and hence $f''(a) = 1 \quad 2\,b_2 = 2!\,b_2$ so that $b_2 = \frac{f''(a)}{2!}$, as indicated in (2). Continuing this process, we obtain $b_n = \frac{f^{(n)}(a)}{n!}$, with which the proof is complete.

Remarks. The series (2) pertaining to a given function $f(x)$ is known as the **Taylor's series*** for $f(x)$ "corresponding to the value (or point) $x = a$." Taylor's series evidently includes Maclaurin's series as a special case; namely, the case in which a is chosen equal to zero. Just as Maclaurin's series is adapted to the computation of $f(x)$ when x has values near to zero (the series then converging with relatively great rapidity), likewise Taylor's series is adapted to computing the values of $f(x)$ when x takes values nearly equal to a, whatever the (fixed) value a may be.

* *Brook Taylor* (1685–1731) was a contemporary of Sir Isaac Newton.

Example. Expand the function $\log x$ in ascending powers of $x - 1$.

SOLUTION. Here $a = 1$ and we have

$$f(x) = \log x, \qquad f(a) = f(1) = \log 1 = 0,$$

$$f'(x) = \frac{1}{x}, \qquad f'(a) = f'(1) = \frac{1}{1} = 1,$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(a) = f''(1) = -\frac{1}{1} = -1,$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(a) = f'''(1) = \frac{2}{1} = 2, \text{ etc.}$$

Hence, substituting in (2), we obtain as the desired series expansion

$$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 +$$

It is readily shown by the theorem of Art. 84 that this series converges for all values of x lying between 0 and 2. It converges rapidly for values of x near to 1 and hence it can be used, if desired, to calculate to a high degree of accuracy the logarithms of numbers which differ but slightly from 1.

91. Second form of Taylor's series. Instead of developing a function $f(x)$ in powers of x or of $x - a$, thus leading the two types of series discussed in Arts. 89, 90, it is frequently desirable to develop a function of the sum of two quantities, as $f(x + h)$, in powers of the added number, h. The form of such a development is directly obtainable from the theorem of Art. 90. Thus, if in that theorem we take $a = h$ and then replace x by $x + h$, we obtain

$$f(x+h) = f(h) + \frac{f'(h)}{1!}x + \frac{f''(h)}{2!}x^2 + \frac{f'''(h)}{3!}x^3 + \\ + \frac{f^{(n)}(h)}{n!}x^n +$$

If we now interchange x and h in this result, noting that the left member remains unchanged since $f(h + x) \equiv f(x + h)$, we obtain

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots \\ + \frac{f^{(n)}(x)}{n!}h^n + \cdots. \qquad (1)$$

This evidently expresses $f(x + h)$ in the form proposed; that is, in powers of the added number h.

Inasmuch as the foregoing formula for $f(x + h)$ is obtainable, as we have shown, from (2) of Art. 90 by a mere change of letters, the two formulas are in reality equivalent. Thus, the series (1) just obtained is to be regarded as a second form of Taylor's series.

Example. Expand $\sin(x+h)$ in ascending powers of h.

SOLUTION. Here $f(x+h) = \sin(x+h)$, $f(x) = \sin x$, and therefore $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, etc. Substituting these expressions in (1), we obtain as the desired development

$$\sin(x+h) = \sin x + \frac{\cos x}{1!}h - \frac{\sin x}{2!}h^2 - \frac{\cos x}{3!}h^3 + \cdots.$$

92. Taylor's series for a function of two or more variables. It is desirable to note the following theorem by means of which, if a function $f(x, y)$ of the two independent variables x, y be given, one may develop the expression $f(x+h, y+k)$ in a series containing the ascending powers of h and k (compare (1), Art. 91):

Theorem. *Given a function $f(x, y)$ of the two independent variables x, y. If the expression $f(x+h, y+k)$ is then developable in ascending powers of h and k (including the products of any two such powers), the development must necessarily have the form*

$$\begin{aligned} f(x+h, y+k) = f(x, y) &+ \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) \\ &+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) \\ + \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} \right. &\left. + 3hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \cdots, \end{aligned} \qquad (1)$$

where the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\,\partial y}, \frac{\partial^2 f}{\partial y^2}$, are assumed to exist and be finite at the special point (x, y) under consideration.

PROOF. The values of x, y, h, and k being regarded as already assigned and hence as fixed, consider the function of the single variable t defined as follows:

$$F(t) = f(x+ht, y+kt). \qquad (2)$$

For the special value $t = 1$ this function evidently reduces to $f(x+h, y+k)$; that is, we have

$$F(1) = f(x+h, y+k). \qquad (3)$$

Moreover, expanding $F(t)$ in Maclaurin's series (Art. 89), we obtain

$$F(t) = F(0) + \frac{F'(0)}{1!}t + \frac{F''(0)}{2!}t^2 + \frac{F'''(0)}{3!}t^3 + \qquad (4)$$

Here $F(0) = f(x, y)$ from (2) and we may likewise express $F'(0)$, $F''(0)$, $F'''(0)$, etc., in terms of x and y, as will now be shown.

Let $$\alpha = x + ht, \quad \beta = y + kt. \tag{5}$$

Then, by the theorem of Art. 63, we may write

$$F'(t) = \frac{\partial F}{\partial \alpha}\frac{d\alpha}{dt} + \frac{\partial F}{\partial \beta}\frac{d\beta}{dt}. \tag{6}$$

But from (4) we have

$$\frac{d\alpha}{dt} = h \text{ and } \frac{d\beta}{dt} = k. \tag{7}$$

Moreover, $F(t)$ may be looked upon as a function of x where x itself is a function of α through (5). Likewise, $F(t)$ may be looked upon as a function of y where y itself is a function of β through (5). Hence, employing the theorem of Art. 16, we may write

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \alpha}\frac{d\alpha}{dx} \text{ and } \frac{\partial F}{\partial y} = \frac{\partial F}{\partial \beta}\cdot\frac{d\beta}{dy}. \tag{8}$$

Here, as a result of (5) we have $\frac{d\alpha}{dx} = 1$ and $\frac{d\beta}{dy} = 1$, so that (8) becomes

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \alpha} \text{ and } \frac{\partial F}{\partial y} = \frac{\partial F}{\partial \beta}. \tag{9}$$

Substituting in (6) the values of $\frac{d\alpha}{dt}, \frac{d\beta}{dt}, \frac{\partial F}{\partial \alpha}, \frac{\partial F}{\partial \beta}$ obtained in (7) and (9), we obtain

$$F'(t) = h\frac{\partial F}{\partial x} + k\frac{\partial F}{\partial y}. \tag{10}$$

Replacing $F(t)$ by $F'(t)$ in (10), we have

$$\begin{aligned} F''(t) &= h\frac{\partial F'}{\partial x} + k\frac{\partial F'}{\partial y} \\ &= h\left\{h\frac{\partial^2 F}{\partial x^2} + k\frac{\partial^2 F}{\partial x\,\partial y}\right\} + k\left\{h\frac{\partial^2 F}{\partial x\,\partial y} + k\frac{\partial^2 F}{\partial y^2}\right\}, \end{aligned}$$

which reduces to

$$F''(t) = h^2\frac{\partial^2 F}{\partial x^2} + 2\,hk\frac{\partial^2 F}{\partial x\,\partial y} + k^2\frac{\partial^2 F}{\partial y^2}. \tag{11}$$

In like manner we may obtain the following form for $F'''(t)$:

$$F'''(t) = h^3\frac{\partial^3 F}{\partial x^3} + 3\,h^2k\frac{\partial^3 F}{\partial x^2\,\partial y} + 3\,hk^2\frac{\partial^3 F}{\partial x\,\partial y^2} + k^3\frac{\partial^3 F}{\partial y^3}. \tag{12}$$

We have now but to place $t = 0$ in (10), (11), (12) to obtain

$$F'(0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y},$$

$$F''(0) = h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2},$$

$$F'''(0) = h^3\frac{\partial^3 f}{\partial x^3} + 3\,h^2k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3\,hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3},$$

etc.

Substituting these expressions in (3), recalling also that $F(0) = f(x, y)$, we obtain

$$F(t) = f(x, y) + t\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$
$$+ \frac{t^2}{2\,!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$
$$+ \frac{t^3}{3\,!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3\,h^2k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3\,hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \cdots.$$

Hence, placing $t = 1$ and recalling formula (2), we arrive at the result stated in the theorem.

Remarks. In the case of a function of three variables, as $f(x, y, z)$, the series corresponding to (1) is found in a similar manner to be

$$f(x + h, y + k, z + l) = f(x, y, z) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z}$$
$$+ \frac{1}{2\,!}\left(h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + l^2\frac{\partial^2 f}{\partial z^2} + 2\,hk\frac{\partial^2 f}{\partial x\,\partial y} + 2\,hl\frac{\partial^2 f}{\partial z\,\partial x} + 2\,kl\frac{\partial^2 f}{\partial y\,\partial z}\right)$$
$$+ \cdots. \qquad (13)$$

More generally, the corresponding series for a function of any number of variables may be written out. As regards the convergence of such series, it is shown in more advanced texts that, in general, every series (1) converges to the value $f(x + h, y + k)$, provided that values sufficiently near to zero are assigned to the quantities h and k. Similar remarks apply to series of the type (13), provided that values sufficiently near to zero are assigned to h, k, and l. Moreover, the nearer the values of h, k, etc. are to zero, the more rapid the convergence.

Example. Having given the function $f(x, y) = \sin(x + y)$, develop $f(x + h, y + k)$ in powers of h and k.

SOLUTION. Here we have

$$f(x, y) = \sin(x + y),$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \cos(x + y),$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y^2} = -\sin(x + y),$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2\,\partial y} = \frac{\partial^3 f}{\partial x\,\partial y^2} = \frac{\partial^3 f}{\partial y^3} = -\cos(x + y),$$

Hence, applying the theorem, the desired development is seen to be

$$f(x + h, y + k) = \sin(x + y) + (h + k)\cos(x + y)$$
$$-\frac{1}{2!}(h^2 + 2hk + k^2)\sin(x + y) - \frac{1}{3!}(h^3 + 3h^2k + 3hk^2 + k^3)\cos(x + y) +$$

93. The theory of approximation. In the formula (1) of Art. 91 let us replace h by Δx, thus writing the formula in the form

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\overline{\Delta x}^2 + \frac{f'''(x)}{3!}\overline{\Delta x}^3 + \qquad (1)$$

If, now, Δx is small in absolute value (though not exactly equal to zero), the terms on the right in (1) that contain Δx, $\overline{\Delta x}^2$, $\overline{\Delta x}^3$, etc. are evidently all small, but of these terms the one containing $\overline{\Delta x}^2$ is smaller than the one containing Δx, the one containing $\overline{\Delta x}^3$ is smaller than the one containing $\overline{\Delta x}^2$, etc. In using formula (1), therefore, to obtain the value of $f(x + \Delta x)$ when Δx is small in absolute value, there will be but a comparatively small error committed if all the terms on the right following the term $f'(x)\Delta x$ are entirely neglected. Otherwise stated, this result may be summarized as follows:

Theorem. *If Δx is small in absolute value, an approximate value of $f(x + \Delta x)$ is given by the relation*

$$f(x + \Delta x) = f(x) + f'(x)\,\Delta x. \qquad (2)$$

The expression here appearing on the right is called the **first approximation** for $f(x + \Delta x)$. If greater accuracy is desired, one may use the so-called **second approximation,** which differs from the first only in that the term $\frac{f''(x)}{2}\overline{\Delta x}^2$ appearing in (1) is retained instead of

neglected; that is, the second approximation for $f(x + \Delta x)$ is given by the formula

$$f(x + \Delta x) = f(x) + f'(x)\,\Delta x + \frac{f''(x)}{2!}\,\overline{\Delta x}^2.$$

Similarly, the third and higher approximations for $f(x + \Delta x)$ are immediately obtainable from (1) by retaining more and more of the neglected terms.

In an analogous manner, one may obtain from the results of Art. 92 first, second, and higher approximation formulas for $f(x + \Delta x,\ y + \Delta y)$, $f(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$, etc., where $f(x, y)$, $f(x, y, z)$, etc. are respectively functions of two, three, or more independent variables. Thus, as a first approximation for $f(x + \Delta x,\ y + \Delta y)$ we have

$$f(x + \Delta x,\ y + \Delta y) = f(x, y) + \frac{\partial f}{\partial x}\,\Delta x + \frac{\partial f}{\partial y}\,\Delta y. \qquad (3)$$

Example 1. If each edge of a cube be increased by one tenth of an inch, find the approximate volume of the new cube thus formed, provided that the original cube had an edge of (*a*) 10 inches; (*b*) 20 inches.

SOLUTION. Let $V(x)$ represent the volume of the cube whose edge is x. Then we have

$$V(x) = x^3.$$

An approximate value for $V(x + \Delta x)$ as determined by (2) is therefore

$$V(x + \Delta x) = x^3 + 3\,x^2\,\Delta x,$$

it being understood that Δx is confined to small values.

In case (*a*) we have $x = 10$ and $\Delta x = 0.1$, so that the desired approximation to the volume of the new cube may be taken as

$$V(10 + 0.1) = 10^3 + 3 \cdot 10^2 \cdot 0.1 = 1030 \text{ cu. in.}$$

Inasmuch as the *exact* volume of the new cube is $(10.1)^3 = 1030.301$ cu. in., it appears that the approximation just determined differs from the true amount by as little as 0.301 cu. in.

In case (*b*) we have $x = 20$ and $x = 0.1$, so that the desired approximation to the volume of the new cube may be taken as

$$V(20 + 0.1) = 20^3 + 3 \quad 20^2 \cdot 0.1 = 8120 \text{ cu. in.}$$

Here the *exact* volume of the new cube is $(20.1)^3 = 8120.601$ cu. in., so that the approximation just obtained differs from the true amount by as little as 0.601 cu. in.

EXERCISES

1. Obtain each of the following Taylor series developments:

(*a*) $e^x = e\left[1 + (x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \cdots\right];$

(*b*) $e^x = e^2\left[1 + (x-2) + \frac{1}{2!}(x-2)^2 + \frac{1}{3!}(x-2)^3 + \quad\right];$

(*c*) $\log x = \log 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 -$

(*d*) $\sin x = \frac{1}{2}\sqrt{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \quad\right];$

(*e*) $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{2^3}(x-1)^2 + \frac{1}{2^4}(x-1)^3 - \frac{5}{2^7}(x-1)^4 +$

(*f*) $x^n = a^n + na^{n-1}(x-a) + \frac{n(n-1)}{2!}a^{n-2}(x-a)^2 +$

(*g*) $\log x = \log a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \frac{(x-a)^4}{4a^4} +$

2. Obtain each of the following developments:

(*a*) $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^3 +$

(*b*) $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} +$

(*c*) $\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x +$

(*d*) $\tan(x+h) = \tan x + h\sec^2 x + h^2\sec^2 x\tan x$
$$+ \frac{h^3}{3}\sec^2 x(1 + 3\tan^2 x) +$$

(*e*) $\log\sin(x+h) = \log\sin x + h\cot x - \frac{h^2}{2}\csc^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots;$

(*f*) $\log\sec(x+h) = \log\sec x + h\tan x + \frac{h^2}{2}\sec^2 x$
$$+ \frac{h^3}{3}\sec^2 x\tan x + \frac{h^4}{12}\sec^2 x(1 + 3\tan^2 x) +$$

3. Prove that, if $P(x)$ is any polynomial of the nth degree in x, it may be expressed as the sum of $n + 1$ terms containing the ascending powers of $x - a$, where a is any given number, as follows:

$$P(x) = P(a) + P'(a)(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \quad + \frac{P^{(n)}(a)}{n!}(x-a)^n.$$

4. If $f(x, y) = Ax^2 + Bxy + Cy^2$, show that

$$f(x+h, y+k) = Ax^2 + Bxy + Cy^2 + (2Ax + By)h + (Bx + 2Cy)k + Ah^2 + Bhk + Ck^2.$$

5. If $f(x, y) = \sin xy$, show that

$$f(x+h, y+k) = \sin xy + (hy + kx)\cos xy - \tfrac{1}{2}(hy + kx)^2\sin xy + \cdots.$$

6. Show that when Δx is small in absolute value, we have approximately

$$\sin(x + \Delta x) = \sin x + \Delta x \quad \cos x,$$

$$\cos(x + \Delta x) = \cos x - \Delta x \quad \sin x.$$

Hence, compute approximate values for sin 46° and cos 46°, it being given that $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} = 0.70711^{+}$ and that $1° = \frac{\pi}{180}$ radians $= 0.01745^{+}$ radians. Compare your results with those given in trigonometric tables for sin 46° and cos 46°

7. Find by the theorem of Art. 93 the approximate value of $\sqrt[5]{32.16}$.

HINT. Apply the theorem to $(x + \Delta x)^{\frac{1}{5}}$, subsequently taking $x = 32$, $\Delta x = 0.16$.

8. Find by the theorem of Art. 93 approximate values for each of the following:

(a) $\sqrt[3]{8.04}$; (b) $\sqrt[4]{16.16}$; (c) $(81.08)^{\frac{3}{4}}$.

9. Show that the approximate value of $e^{.97}$ is 2.6380.

10. Prove that so long as x and Δx have any values for which $\left|\frac{\Delta x}{x}\right| < 1$ the error in the approximate equation

$$\frac{1}{x + \Delta x} = \frac{1}{x} - \frac{\Delta x}{x^2}$$

will be less than $\frac{\overline{\Delta x}^2}{|x^3|}$.

11. Use the result in Ex. 10 to find how much must be taken from the reciprocal of 4000 in order to give the reciprocal of 4001.

12. How much should be added to 10^4 in order to give $10^{4.001}$?

13. Show that when $\frac{b}{a}$ is small, the value of $\sqrt{a^2 + b}$ is approximately equal to $a + \left(\frac{b}{2a}\right)$.

14. Let r be the radius of the base of a right circular cylinder, and let h be the altitude. Show by (3) of Art. 93 that if r and h be increased by the slight amounts a and b respectively, then

(a) the corresponding increase in the volume will be approximately $\pi r(2ah + br)$,

(b) the increase in the lateral area will be approximately $2\pi(ah + br)$,

(c) the increase in the total area will be approximately $2\pi[(2r + h)a + br]$.

15. Let r be the radius of the base of a right circular cone, and let h be the altitude. Show that if r and h be increased by the slight amounts a and b respectively, then the corresponding increase in the lateral area of the cone will be approximately

$$\frac{(2r^2 + h^2)a + rhb}{\sqrt{a^2 + r^2}}.$$

CHAPTER XI

INFINITESIMALS. DIFFERENTIALS

94. Definition. The term **infinitesimal** is used in the calculus to denote any expression which, in the course of a given discussion, approaches the limit zero. For example, the expression $x - 1$ is an infinitesimal in any discussion wherein x itself is approaching the limit 1, for under such circumstances the difference $x - 1$ is obviously approaching the limit zero. Similarly, the expressions $(x - 1)^2$, $x(x - 1)$, $\sin (x - 1)$, $\log x$ are infinitesimals in any discussion in which x is approaching the limit 1. Again, the expressions x^2, $\sin x$, $x + 2x^2$, $e^x - 1$ are infinitesimals in any discussion wherein x is itself approaching the limit zero. Geometrical illustrations of infinitesimals occur in Fig. 102, where, in case the central angle x be allowed to approach zero, the arc AP becomes an infinitesimal, as likewise also the chord AP and each of the lengths MP, AQ, PN, MA, PQ, PR, and AR. On the other hand, OM is obviously not an infinitesimal in any discussion in which x itself approaches the limit zero, though it might be in some other discussion, as, for example, one wherein x is approaching the limit $\frac{\pi}{2}\cdot$

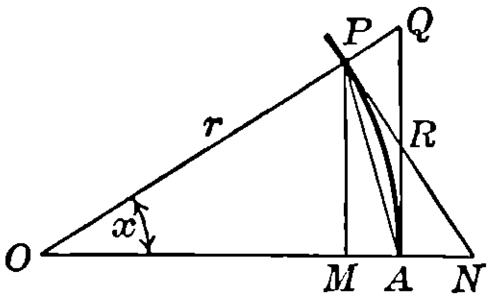

Fig. 102

95. Comparison of infinitesimals. Suppose that in a certain discussion two infinitesimals are present. Let them be represented by $\epsilon_1(x)$ and $\epsilon_2(x)$,* each being dependent upon some ultimate variable x and each approaching zero as this variable x itself approaches some given fixed value which we shall suppose to be $x = a$.

For example, in Fig. 102 we have (in particular) the two infinitesimals $\epsilon_1(x) = MP$ and $\epsilon_2(x) = AQ$, each being dependent upon the central angle x and each approaching zero as this angle approaches that value.

We then have the following definitions:

* The Greek letter ϵ (pronounced ĕp'si-lon) is commonly used to represent an infinitesimal.

Definitions. *(a) The infinitesimal* $\epsilon_2(x)$ *is said to be of the same order as the infinitesimal* $\epsilon_1(x)$ *in case the limit of the ratio* $\frac{\epsilon_2(x)}{\epsilon_1(x)}$ *as* x *approaches* a *is some number different from zero; that is, in case*

$$\lim_{x \to a} \frac{\epsilon_2(x)}{\epsilon_1(x)} = K \neq 0.$$

(b) The infinitesimal $\epsilon_2(x)$ *is said to be of higher order than the infinitesimal* $\epsilon_1(x)$ *in case the limit just mentioned has the value zero; that is, in case*

$$\lim_{x \to a} \frac{\epsilon_2(x)}{\epsilon_1(x)} = 0.$$

(c) The infinitesimal $\epsilon_2(x)$ *is said to be of lower order than the infinitesimal* $\epsilon_1(x)$ *in case the limit just mentioned is* $+\infty$ *or* $-\infty$ *that is, in case*

$$\lim_{x \to a} \frac{\epsilon_2(x)}{\epsilon_1(x)} = \pm\infty.$$

Thus in Fig. 102 (or 103) the line AQ is an infinitesimal of the same order as the line MP since by trigonometry we have $AQ = r \tan x$ and $MP = r \sin x$, so that

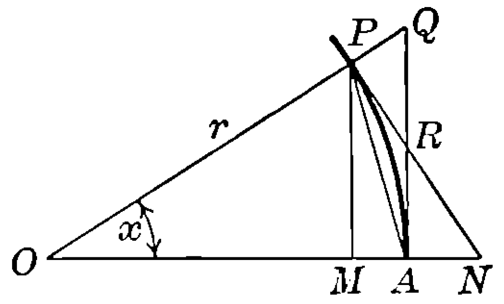

FIG. 103

$$\frac{AQ}{MP} = \frac{r \tan x}{r \sin x} = \frac{\tan x}{\sin x} = \sec x$$

and hence

$$\lim_{x \to 0} \frac{AQ}{MP} = \lim_{x \to 0} \sec x = 1 \neq 0.$$

On the other hand, we may show as follows that MA is of higher order than MP. We have, in fact,

$$\frac{MA}{MP} = \frac{OA - OM}{MP} = \frac{r - r\cos x}{r \sin x} = \frac{1 - \cos x}{\sin x}. \tag{1}$$

Here, if we allow x to approach the limit zero, the right member assumes in the limit the indeterminate form $\frac{0}{0}$, but its value may be determined by use of the theorem of Art. 74. Thus, we obtain

$$\lim_{x \to 0} \frac{MA}{MP} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0. \tag{2}$$

Inasmuch as the value of the left member of (2) is thus found to be zero, definition (*b*) applies; that is, MA is of higher order than MP.

Besides the definitions (*a*), (*b*), (*c*), which serve the general purpose of determining whether two infinitesimals are of the same order, and if not which one is of the higher order, we have also the following definition through which the orders of any two infinitesimals may be more specifically compared:

(*d*) *The infinitesimal* $\epsilon_2(x)$ *is said to be of the nth order as compared to the infinitesimal* $\epsilon_1(x)$ *in case the limit of the ratio* $\frac{\epsilon_2(x)}{[\epsilon_1(x)]^n}$ *as* x *approaches a is some number different from zero; that is, in case*

$$\lim_{x \to a} \frac{\epsilon_2(x)}{[\epsilon_1(x)]^n} = K \neq 0.$$

Example 1. Show that as x approaches zero in Fig. 103 the length MA is an infinitesimal of the second order as compared to the length MP.

SOLUTION. Reasoning as in obtaining (1), we may write

$$\frac{MA}{\overline{MP}^2} = \frac{OA - OM}{\overline{MP}^2} = \frac{r - r\cos x}{r^2 \sin^2 x} = \frac{1 - \cos x}{r \sin^2 x},$$

and taking the limit of each member of this equation as x approaches zero, evaluating the resulting indeterminate form on the right by means of the theorem of Art. 74, we obtain

$$\lim_{x \to 0} \frac{MA}{\overline{MP}^2} = \lim_{x \to 0} \frac{1 - \cos x}{r \sin^2 x} = \frac{0}{0} = \lim_{x \to 0} \frac{\sin x}{2\,r \sin x \cos x} = \lim_{x \to 0} \frac{1}{2\,r \cos x} = \frac{1}{2\,r}.$$

This result, being the number $\frac{1}{2\,r}$ which is different from zero, indicates that definition (*d*) applies, and it shows, as stated before, that MA is of the *second* order as compared to MP.

Example 2. Show that as x approaches zero in Fig. 103 the length MA is an infinitesimal of the first order as compared to the length PQ.

SOLUTION. Reasoning as before, we may write

$$\frac{MA}{PQ} = \frac{OA - OM}{OQ - OP} = \frac{r - r\cos x}{r \sec x - r} = \frac{(1 - \cos x)\cos x}{1 - \cos x} = \cos x.$$

Hence,

$$\lim_{x \to 0} \frac{MA}{PQ} = \lim_{x \to 0} \cos x = 1 \neq 0.$$

By definition (*d*), MA is thus of the first order as compared to PQ.

Example 3. Show that as x approaches the value $\frac{\pi}{2}$ the expression $\sin 2x \cdot \log \sin x$ becomes an infinitesimal of the third order as compared to the expression $\pi - 2x$.

SOLUTION. What we are to show, when stated as an equation, is that

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin 2x \cdot \log \sin x}{(\pi - 2x)^3} = K = \textit{some number} \neq 0. \tag{3}$$

Now, we may write

$$\frac{\sin 2x \cdot \log \sin x}{(\pi - 2x)^3} = \frac{\sin 2x}{\pi - 2x} \quad \frac{\log \sin x}{(\pi - 2x)^2}$$

and hence

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin 2x \quad \log \sin x}{(\pi - 2x)^3} = \lim_{x \to \frac{\pi}{2}} \frac{\sin 2x}{\pi - 2x} \quad \lim_{x \to \frac{\pi}{2}} \frac{\log \sin x}{(\pi - 2x)^2}.$$

But, applying the theorem of Art. 74, we have

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin 2x}{\pi - 2x} = \frac{0}{0} = \lim_{x \to \frac{\pi}{2}} \frac{2 \cos 2x}{-2} = -\cos \pi = +1$$

and

$$\lim_{x \to \frac{\pi}{2}} \frac{\log \sin x}{(\pi - 2x)^2} = \frac{0}{0} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-4(\pi - 2x)} = -\frac{1}{4} \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{(\pi - 2x) \sin x} = \frac{0}{0}$$

$$= -\frac{1}{4} \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{(\pi - 2x) \cos x - 2 \sin x} = -\frac{1}{4} \quad \frac{-1}{-2} = -\frac{1}{8}.$$

The desired result (3) thereby becomes established, the value of K being $-\frac{1}{8}$.

96. Order of contact of two curves. Suppose that C_1 and C_2 are two curves which are tangent to each other at a certain ordinary point P; that is, have one and only one common tangent line PT at P Now let a point M be selected on PT near to P and let the perpendicular MN be drawn intersecting C_1 and C_2 at P_1 and P_2 respectively. Consider the distance P_1P_2 along this perpendicular.

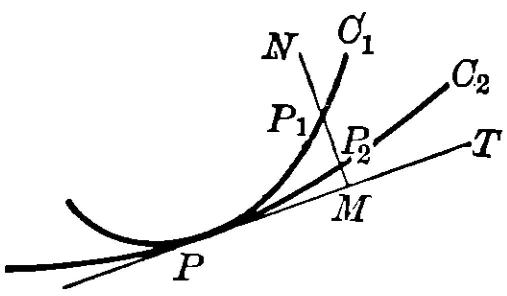

FIG. 104

Evidently it approaches zero as the point M is chosen nearer and nearer to P In other words, P_1P_2 and PM are infinitesimals as M approaches P. Under these conditions we have the following definition:

Definition. *If, as M approaches P along the tangent at P, the distance P_1P_2 is an infinitesimal of the nth order as compared to the infinitesimal PM, the two curves C_1 and C_2 are said to have* ***contact of the (n − 1)st order*** *at P.*

Thus, it is shown in more advanced texts that a curve C_1 will ordinarily have contact of the *first* order with its own tangent line PT (regarded as the second curve C_2); that is, at an ordinary point P on a curve C_1 the distance P_1M is of the second order of infinitesimal as compared to PM. This is illustrated in Ex. 1 below. In exceptional cases, however (as at a point of inflection), P_1M is of the third, or even higher order of infinitesimal as compared to PM.

Again, it is shown in advanced calculus that the circle of curvature (see Art. 50) of a curve C at a specified point P has contact of the *second* or higher order with the curve itself at this point; that is, using Fig. 105, P_1P_2 is an infinitesimal of at least the third order as compared to PM.

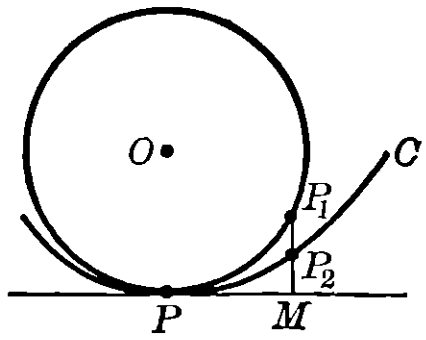

FIG. 105

The above facts from advanced calculus are here mentioned without proof in order that the student may perceive at once the intimate connection between the subject of infinitesimals and that of geometry. That infinitesimals are likewise connected with other important aspects of mathematics will be evident from what follows in this and succeeding chapters.

Example 1. Show that the parabola $y = 4x^2$ has contact of the first order with the tangent at its vertex.

SOLUTION. The equation of the tangent at the vertex $P \equiv (0, 0)$ is found at once by the theorem of Art. 45 to be $y = 0$. Thus, the tangent in question coincides with the X-axis. If, therefore, a point P_1 be chosen on the tangent near to P at the distance x from it, and if the perpendicular P_1N be then drawn intersecting the parabola at P_2, we have $PP_1 = x$, $P_1P_2 = 4x^2$. Hence,

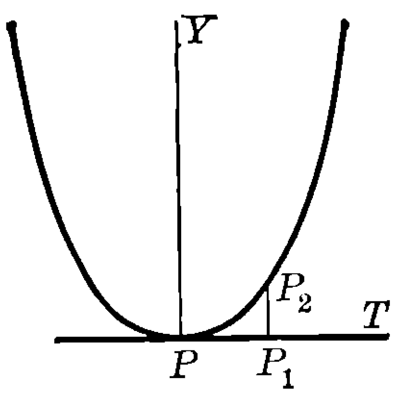

FIG. 106

$$\lim_{x \to 0} \frac{P_1P_2}{\overline{PP_1}^2} = \lim_{x \to 0} \frac{4x^2}{x^2} = \lim_{x \to 0} 4 = 4 \neq 0.$$

Therefore, as P_1 approaches P the distance P_1P_2 is an infinitesimal of the second order as compared to PP_1. By the foregoing definition, therefore, the parabola has contact of the first order with its tangent at the point P in question.

Example 2. Show that the parabola $y = 1 - \frac{1}{2}x^2$ has contact of the third order with the curve $y = \cos x$ at the point (0, 1).

SOLUTION. The equation of the common tangent PT to the two curves at the point (0, 1) is readily found to be

$$y = 1.$$

If now a point M be chosen upon PT near to P at a distance x from it and if the perpendicular MN be then drawn to PT intersecting the two curves at the points P_1 and P_2 respectively, we have $PM = ON = x$ and

$$P_1P_2 = P_1M - P_2M = NP_2 - NP_1 = \cos x - (1 - \tfrac{1}{2}x^2).$$

Hence, by repeated use of the theorem of Art. 74, we obtain

$$\lim_{x \to 0} \frac{P_1P_2}{\overline{PM}^4} = \lim_{x \to 0} \frac{\cos x - (1 - \frac{1}{2}x^2)}{x^4} = \frac{0}{0} = \lim_{x \to 0} \frac{-\sin x + x}{4x^3} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-\cos x + 1}{12x^2} = \frac{0}{0} = \lim_{x \to 0} \frac{\sin x}{24x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{24} = \frac{1}{24} \neq 0.$$

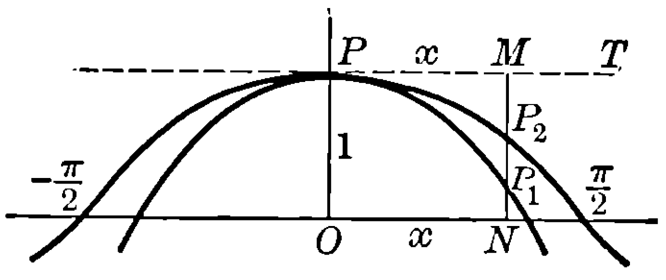

FIG. 107

Inasmuch as P_1P_2 is thus an infinitesimal of the fourth order as compared to PM, it follows by the foregoing definition that the two curves have contact of the third order at the point (0, 1) in question.

EXERCISES

1. State why each of the following expressions is an infinitesimal as x approaches the value 2 and determine for each whether it is of the same order, or of higher order or of lower order than the infinitesimal $(x - 2)$:

(*a*) $(x-2)^2$; (*b*) $x^2 - 4$; (*c*) $x^2 - 3x + 2$; (*d*) $\sin(x-2)$; (*e*) $\sin^2(x-2)$; (*f*) $\sin(x-2)^2$; (*g*) $\tan(x-2)$; (*h*) $\log(x-1)$; (*i*) $\sqrt{x-2}$; (*j*) $\sqrt[3]{x^2-4}$; (*k*) $e^{(x-2)} - 1$.

2. As x approaches zero show that, as compared to x,

(*a*) the infinitesimal $2x - x^2$ is of order 1;

(*b*) the infinitesimal $x^2 - 2x^3 + x^4$ is of order 2;

(*c*) the infinitesimal $3\sqrt{x} + x$ is of order $\frac{1}{2}$;

(*d*) the infinitesimal $\sin x$ is of order 1;

(*e*) the infinitesimal $e^x - 1 - x$ is of order 2;

(*f*) the infinitesimal $2\log(1+x) - 2x + x^2$ is of order 3;

(*g*) the infinitesimal $\sin^2 x - x^2$ is of order 4.

3. As the angle x in Fig. 103 approaches zero show that

(a) AR is an infinitesimal of the first order as compared to x;

(b) AN is an infinitesimal of higher order than RQ;

(c) AR and RN are infinitesimals of the same order;

(d) PQ is of the second order as compared to x.

4. Determine the order of each of the following infinitesimals as compared to x when x approaches zero:

(a) $\tan x$; (d) $\sin x - x$; (g) $\text{arc} \sin x - x$;

(b) $e^x - 1$; (e) $\cos x - 1 - \frac{x^2}{2}$; (h) $\text{arc} \tan x - x$;

(c) $\log(1+x) - x$; (f) $\cos^2 x - 1$; (i) $e^x \sin x - x$.

5. Show that the circle $x^2 + (y - 1)^2 = 1$ has contact of the first order with its tangent at the origin. Draw a figure.

6. Show that the circle of Ex. **5** has contact of the first order with its tangent at the point (1, 1). Draw a figure.

7. Show that the cubical parabola $y = x^3$ has contact of the second order with its tangent at the origin. Draw a figure.

8. Show that the curve mentioned in Ex. 7 has contact of the third order at the origin with the curve $y = x^3 - x^4$. Draw a figure.

9. Show that the parabola $y = \frac{1}{4} x^2$ has contact of the third order at the origin with its circle of curvature at that point.

HINT. Show first that the radius of curvature of the given parabola at the origin is $R = 2$; hence that the equation of the circle of curvature at this point is

$$x^2 + (y - 2)^2 = 2^2 = 4.$$

From this equation we have $y - 2 = \pm\sqrt{4 - x^2}$ so that the smaller of the two values of y on the circle corresponding to any given value of x is $y = 2 - \sqrt{4 - x^2}$.

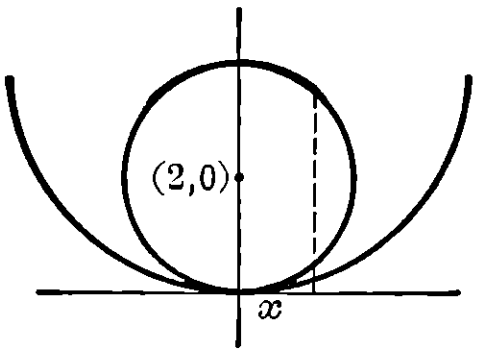

FIG. 108

The radical here appearing may now be expanded into a Maclaurin's series (see Ex. 4, page 181), so that the value of y just mentioned may be put into the following form, at least when x is small in absolute value,

$$y = 2 - \sqrt{4 - x^2} = 2 - 2\left(1 - \frac{x^2}{4}\right)^{\frac{1}{2}} = 2 - 2\left[1 - \frac{1}{2}\left(\frac{x^2}{4}\right) - \frac{1}{8}\left(\frac{x^2}{4}\right)^2 + \quad\right]$$
$$= \frac{x^2}{4} + \frac{1}{4}\left(\frac{x^2}{4}\right)^2 +$$

10. Show that the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has contact of the third order at the upper end of its minor axis with its circle of curvature at that point.

HINT. First show by means of Ex. 7, page 98, that the radius of the circle of curvature at the point in question is $R = \frac{a^2}{b}$ and hence that the coördinates of its center are $\left(0, \frac{b^2 - a^2}{b}\right)$. Now proceed analogously to the hint of Ex. 9.

97. Further property of infinitesimals. Let $\epsilon_1(x)$ and $\epsilon_2(x)$ be two infinitesimals which are known to be of the same order as x approaches the value a; that is, suppose that $\lim_{x\to a} \epsilon_1(x) = 0$, $\lim_{x\to a} \epsilon_2(x) = 0$, and that

$$\lim_{x\to a} \frac{\epsilon_2(x)}{\epsilon_1(x)} = K \neq 0.$$

We now note that the particular case in which the value of K is 1 is of especial importance in the calculus. Corresponding to this case, in fact, we have the following general theorem which is of frequent use, as will appear in this and later chapters:

Theorem. *If, as x approaches a, the expressions $\epsilon_1(x)$ and $\epsilon_2(x)$ are two infinitesimals such that*

$$\lim_{x\to a} \frac{\epsilon_2(x)}{\epsilon_1(x)} = 1, \tag{1}$$

then the difference $\epsilon_2(x) - \epsilon_1(x)$ is an infinitesimal of higher order than either $\epsilon_1(x)$ or $\epsilon_2(x)$.

Conversely if as x approaches a, the difference $\epsilon_2(x) - \epsilon_1(x)$ is an infinitesimal of higher order than either $\epsilon_1(x)$ or $\epsilon_2(x)$, then relation (1) is satisfied.

PROOF. If (1) is true, the ratio $\frac{\epsilon_2(x)}{\epsilon_1(x)}$ is necessarily of the form

$$\frac{\epsilon_2(x)}{\epsilon_1(x)} = 1 + \eta(x), \tag{2}$$

where $\eta(x)$* is some expression such that $\lim_{x\to a} \eta(x) = 0$. Upon clearing (2) of fractions and transposing, we obtain

$$\epsilon_2(x) - \epsilon_1(x) = \epsilon_1(x)\eta(x)$$

and therefore we may write

$$\frac{\epsilon_2(x) - \epsilon_1(x)}{\epsilon_1(x)} = \eta(x).$$

It follows that

$$\lim_{x\to a} \frac{\epsilon_2(x) - \epsilon_1(x)}{\epsilon_1(x)} = \lim_{x\to a} \eta(x) = 0.$$

Hence, by definition (*b*) of Art. 95, the difference $\epsilon_2(x) - \epsilon_1(x)$ is an infinitesimal of higher order than $\epsilon_1(x)$. That it is likewise of higher order than $\epsilon_2(x)$ follows from the fact that $\epsilon_2(x)$ and $\epsilon_1(x)$ are themselves of the same order, as follows from (1).

---

* The Greek letters η (pronounced āy'ta) and ξ (pronounced ksee) are frequently used to denote infinitesimals.

Conversely, if $\epsilon_2(x) - \epsilon_1(x)$ is an infinitesimal of higher order than either $\epsilon_1(x)$ or $\epsilon_2(x)$, we have

$$\epsilon_2(x) - \epsilon_1(x) = \xi(x) \tag{3}$$

where

$$\lim_{x\to a} \frac{\xi(x)}{\epsilon_1(x)} = \lim_{x\to a} \frac{\xi(x)}{\epsilon_2(x)} = 0. \tag{4}$$

From (3) we may write

$$\frac{\epsilon_2(x)}{\epsilon_1(x)} = 1 + \frac{\xi(x)}{\epsilon_1(x)},$$

and hence by (4) we have

$$\lim_{x\to a} \frac{\epsilon_2(x)}{\epsilon_1(x)} = \lim_{x\to a}\left(1 + \frac{\xi(x)}{\epsilon_1(x)}\right) = 1 + \lim_{x\to a} \frac{\xi(x)}{\epsilon_1(x)} = 1 + 0 = 1,$$

which is relation (1).

For example, as x approaches $\frac{\pi}{2}$ the expressions $\cos x$ and $\frac{\pi}{2} - x$ are not only both infinitesimals, but we have

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x} = \frac{0}{0} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{-1} = \lim_{x\to\frac{\pi}{2}} \sin x = 1.$$

It follows from the foregoing theorem that as x approaches $\frac{\pi}{2}$ the difference $\left(\frac{\pi}{2} - x\right) - \cos x$ is an infinitesimal of higher order than either $\cos x$ or $\frac{\pi}{2} - x$.

This result is readily interpreted geometrically. Thus, if the curves $y = \cos x$ and $y = \frac{\pi}{2} - x$ are drawn, as in Fig. 109, the difference QR between the ordinates PR and PQ corresponding to the point whose abscissa is x is an infinitesimal of higher order than either PQ or PR as x approaches the value $\frac{\pi}{2}$.

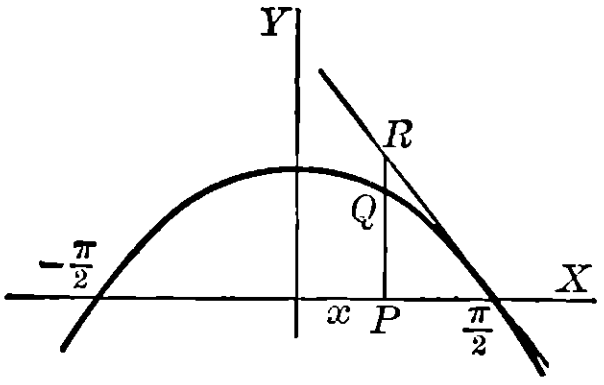

FIG. 109

98. The differential. Although the symbol $\frac{dy}{dx}$ as used throughout the preceding pages to represent the derivative of y with respect to x bears an evident resemblance to a fraction, no attempt has thus far been made to attribute individual meanings to dy and dx. However, such meanings may now be supplied if we are at liberty to use the idea of an infinitesimal as defined in the foregoing articles. To be specific, the function $y = f(x)$ being given and the value of x being regarded for the moment as fixed, let us suppose that an arbitrarily

small increment Δx is given to x. Then Δx is at once an infinitesimal, since its limit is zero, and we define dx to be this infinitesimal; that is, we define

$$dx = \Delta x. \tag{1}$$

The infinitesimal dx as thus defined is known as the **differential of the independent variable x.** As regards dy, which is known as the **differential of the dependent variable y,** it is defined as follows:

$$dy = f'(x) \quad \Delta x, \tag{2}$$

where $f'(x)$ represents the derivative of the given function at the value of x in question.

For example,

$$d(x^2) = 2\,x \cdot \Delta x; \quad d(\sin x) = \cos x \cdot \Delta x, \quad d(\log x) = \frac{\Delta x}{x}.$$

With dx and dy defined as in (1) and (2), the value of y' is at once (as it should be) that of the derivative $f'(x)$. In fact, we have

$$\frac{dy}{dx} = \frac{f'(x) \cdot \Delta x}{\Delta x} = f'(x). \tag{3}$$

It is also to be noted that from (1) and (2) we may write if desired

$$dy = f'(x)\,dx. \tag{4}$$

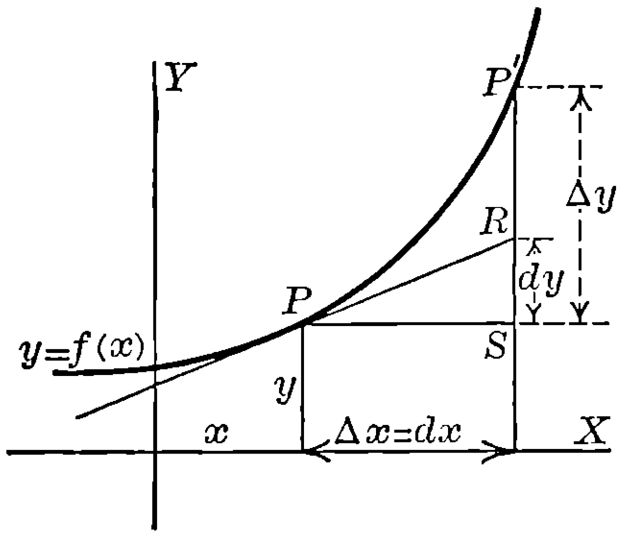

FIG. 110

The geometrical meanings of dx and dy are represented in Fig. 110, that of dy being SR because we have

$$SR = PS \cdot \tan \angle SPR = \Delta x \cdot f'(x) = dy.$$

The figure also shows the geometrical meaning of Δy (compare Fig. 5), where Δy is the increment taken on by y as x changes from x to $x + \Delta x$. Although dy and Δy are not thus the same thing, both are infinitesimals as Δx approaches zero. The relation between them will be stated more precisely in the next article.

Note. It is important to appreciate the fact that, in accordance with the definitions (1) and (2), the differentials dx and dy can never be regarded as having fixed values. Each depends, in fact, upon the assignment made to the arbitrarily small and hence variable quantity (increment) Δx. However, the ratio y' does not contain Δx (see (3)) but has a value which depends, as it should, only upon the value of the independent variable x. Historically, attempts were made in the early development of the calculus to assign fixed values to dx and dy, but such attempts were lacking in mathematical rigor and hence were necessarily abandoned in the later and more critical development of the subject.

99. The relation between dy and Δy. We may now establish the following theorem:

Theorem. *The infinitesimals dy and Δy differ from each other by an infinitesimal of higher order than either of them.*

Thus, in Fig. 110, the difference RP' between SP' and SR is of higher order of infinitesimal as Δx approaches zero than either SP' or SR themselves.

PROOF. If, in the formula of Art. 72, we place $a = x$ and $b = x+\Delta x$, it appears that we may write

$$f(x + \Delta x) = f(x) + \Delta x \cdot f'(x_1),$$

where x_1 is some determinate value lying between x and $x + \Delta x$. Hence, recalling the meaning of Δy, we have

$$\Delta y = f(x + \Delta x) - f(x) = \Delta x \cdot f'(x_1) = f'(x_1) \cdot \Delta x.$$

Therefore, recalling the definition of dy, we have

$$\frac{dy}{\Delta y} = \frac{f'(x)\,\Delta x}{f'(x_1)\,\Delta x} = \frac{f'(x)}{f'(x_1)}.$$

Now, as Δx approaches zero, x_1 must approach the limit x since x_1 lies between $x + \Delta x$ and x. Hence, from (1) we have

$$\lim_{\Delta x \to 0} \frac{dy}{\Delta y} = \lim_{\Delta x \to 0} \frac{f'(x)}{f'(x_1)} = \frac{f'(x)}{f'(x)} = 1.$$

It follows by the theorem of Art. 97 that dy and Δy differ from each other by an infinitesimal of higher order than either, as was to be shown.

100. Differentials of higher orders. If $y = f(x)$, the differential dy as defined by (2) of Art. 98 is known as the *first* differential of y. The *second* differential is defined in an analogous manner to be $f''(x)\ \overline{\Delta x}^2$; likewise the *third* differential is $f'''(x)\ \overline{\Delta x}^3$, etc. As thus defined, the second differential when divided by $\overline{dx}^2$ (written dx^2) gives the second derivative of y, as appears at once upon recalling the definition of dx (see (1), Art. 98). Likewise, the third differential when divided by $\overline{dx}^3$ (written dx^3) gives the third derivative, etc. With these definitions for d^2y, d^3y, d^4y, etc. we thus have the following familiar equations preserved wherein individual meanings are now given for the numerator and denominator in each instance (compare Art. 35):

$$\frac{d^2y}{dx^2} = f''(x), \quad \frac{d^3y}{dx^3} = f'''(x), \quad \frac{d^4y}{dx^4} = f^{(\text{IV})}(x), \qquad \frac{d^ny}{dx^n} = f^{(n)}(x).$$

EXERCISES

1. Show by means of the theorem of Art. 97 that as x approaches zero the difference $x - \sin x$ is an infinitesimal of higher order than either x or $\sin x$. Confirm this result independently by use of the Maclaurin series development of $\sin x$ (see Ex. 2 (*c*), page 180). Also, draw a diagram to show the geometrical significance of the result in question.

2. Show that under the same conditions as in Ex. 1 the difference $2x - \sin x$, although an infinitesimal, is not an infinitesimal of higher order than either $2x$ or $\sin x$. Draw a diagram to show the geometrical significance of such a result.

3. Show that as x approaches zero the difference $x - \log(1 + x)$ is an infinitesimal of higher order than either x or $\log(1 + x)$. Confirm this result by means of Ex. 2 (*b*), page 180, and draw a diagram to show the significance of the result in question. Examine the corresponding questions when $x - \log(1 + 2x)$ is used instead of $x - \log(1 + x)$.

4. Show that as x approaches zero the difference between the infinitesimals x and $\tan x$ is of higher order than either of them. Draw a diagram to show the geometrical significance of this result, and compare with Ex. 1.

5. Determine whether, as x approaches zero, the difference between the two infinitesimals $e^x - e^{-x}$, $\sin x$ is of higher order than either of them.

6. Establish (making free use of the formulas in Chap. III) the correctness of the following equations involving differentials:

(*a*) $d(x^n) = nx^{n-1}\,\Delta x$;

(*b*) $d(x^n) = nx^{n-1}\,dx$;

(*c*) $d(\cos x) = -\sin x\,dx$;

(*d*) $d(\text{arc}\sin 2x) = \dfrac{2\,dx}{\sqrt{1 - 4x^2}}$;

(*e*) $d(e^{t^2}) = e^{t^2}\,2t\,dt$;

(*f*) $d(\log az) = \dfrac{dz}{z}$; a = *constant.*

(*g*) $d^2(e^x + e^{-x}) = (e^x + e^{-x})\,dx^2$;

(*h*) $d^3(e^x \log x) = e^x\left(\log x + \dfrac{3}{x} - \dfrac{3}{x^2} + \dfrac{2}{x^3}\right)dx^3$.

7. If $b^2x^2 + a^2y^2 = a^2b^2$, where a and b are constants, show that

(*a*) $dy = -\dfrac{b^2x}{a^2y}\,dx$; (*b*) $d^2y = -\dfrac{b^4}{a^2y^3}\,dx^2$.

8. By means of the definitions of Art. 98 and the formulas of Art. 15 show that, if u and v are any two functions of x, we may write

$$d(u + v) = du + dv,$$

$$d(uv) = u\,dv + v\,du,$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

9. State (orally) each of the theorems of Art. 17–33 in the language of differentials instead of derivatives, making the necessary change in the formula in each instance.

101. Derivatives in polar coördinates. The geometrical interpretations thus far given for the terms " derivative " and " differential " have been based upon the assumption that a rectangular system of coördinates was being used. For example, it was shown in Art. 9 that the derivative y' of a given function $y = f(x)$, when evaluated for any special value of x, is to be interpreted geometrically as the slope of the curve $y = f(x)$ at the point on this curve whose abscissa is the given value of x. When a polar system of coördinates is used, these interpretations naturally undergo a change. In such a system the position of any point P in the plane is determined by knowing its radius vector, designated by ρ,* and its vectorial angle, designated by θ, and the equation of a curve is typified by $\rho = f(\theta)$, where $f(\theta)$ indicates some function of θ. Moreover, it is to be noted that in a polar system of coördinates the terms " subtangent " and " subnormal " are defined as follows (compare Art. 46): Suppose that P is a given point upon the curve $\rho = f(\theta)$. Draw a line NT through the pole perpendicular to the radius vector OP. If PT is the tangent and PN the normal to the curve at P, then OT is defined to be the **polar subtangent** at P and ON is defined to be **polar subnormal** at P.

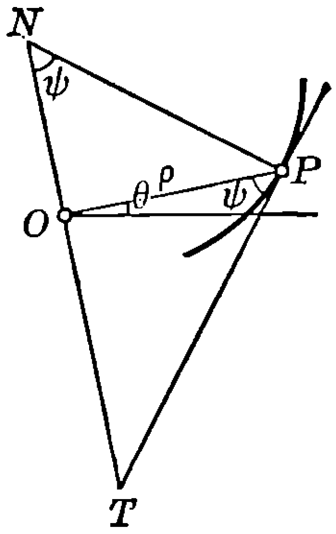

FIG. 111

The foregoing preliminary statements having been made, we may now show that the geometric interpretation of the derivative of a function when polar coördinates are being used is as follows:

Theorem. *The derivative* $\frac{d\rho}{d\theta}$ *of a given function* $\rho = f(\theta)$, *when evaluated for any special value of* θ, *is equal to the subnormal of the curve* $\rho = f(\theta)$ *at the point upon it whose vectorial angle is the given value of* θ. Briefly stated, we have the following formula:

$$\frac{d\rho}{d\theta} = \text{the polar subnormal.}\dagger$$

* The Greek letter ρ is pronounced rō. The Greek letter θ is pronounced thā'ta.

† When ρ increases as θ increases, as at P in Fig. 111, $\frac{d\rho}{d\theta}$ is positive. Then the subnormal ON lies to the left of an observer situated at O and looking along OP. In this case, ON is regarded as positive. If, on the other hand, ρ decreases as θ increases, then $\frac{d\rho}{d\theta}$ is negative, corresponding to which ON lies on the right of an observer situated as just described and is regarded as negative.

PROOF. In Fig. 112 let the coördinates of P be (ρ, θ) and let ψ (pronounced psee) be the angle between the radius vector OP and the tangent PT at P. If θ now takes an increment $\Delta\theta$, ρ will take an increment $\Delta\rho$ represented by $R'Q$, it being understood that PR' is the arc of a circle of radius ρ drawn about O as center. Now draw PR perpendicular to OQ and draw also the secant PQ. In the triangle PQR we may then write

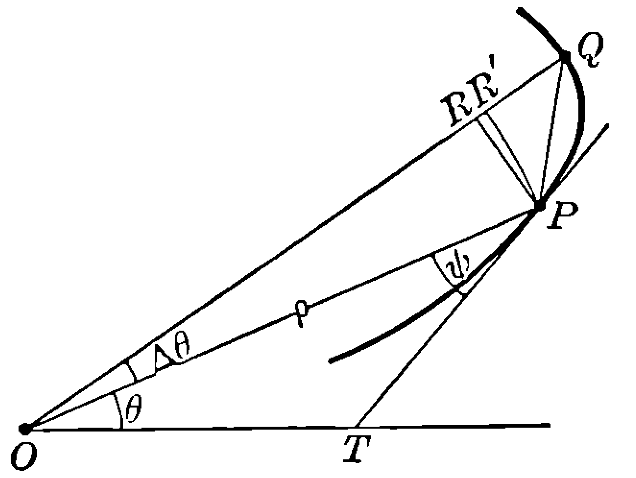

FIG. 112

$$\cot \angle PQR = \frac{RQ}{PR} = \frac{R'Q}{PR'} \cdot \frac{PR'}{PR} \cdot \frac{RQ}{R'Q}.$$

Recalling that $R'Q = \Delta\rho$ and noting that, by elementary mathematics, $PR' = \rho\Delta\theta$, we thus have

$$\cot \angle PQR = \frac{1}{\rho} \cdot \frac{\Delta\rho}{\Delta\theta} \cdot \frac{PR'}{PR} \cdot \frac{RQ}{R'Q}. \tag{1}$$

Now, as $\Delta\theta$ is allowed to approach zero as a limit the quantity $\frac{1}{\rho}$ appearing in (1) remains constant while we have

$$\lim_{\Delta\theta \to 0} \frac{\Delta\rho}{\Delta\theta} = \frac{d\rho}{d\theta},$$

$$\lim_{\Delta\theta \to 0} \frac{PR'}{RP} = 1,$$

$$\lim_{\Delta\theta \to 0} \frac{RQ}{R'Q} = 1.$$

Moreover, as $\Delta\theta$ approaches zero, the angle PQR evidently approaches ψ as its limit. If, therefore, we allow $\Delta\theta$ to approach zero in (1), we obtain in the limit

$$\cot \psi = \frac{1}{\rho} \frac{d\rho}{d\theta}, \tag{2}$$

or

$$\frac{d\rho}{d\theta} = \rho \cot \psi. \tag{3}$$

But, upon referring to Fig. 111, it appears that the right member of (3) is equal to the subnormal ON of P.

The proof is thus complete.

Corollary 1. Corresponding to the foregoing formula for the polar subnormal we have the following for the polar subtangent:

$$\frac{\rho^2}{\frac{d\rho}{d\theta}} = \textit{the polar subtangent.} \tag{4}$$

In fact, from Fig. 111 we may write

$$\begin{aligned} \textit{polar subtangent} &= OT \\ &= \rho \tan \psi = \frac{\rho}{\cot \psi}. \end{aligned}$$

It now suffices to substitute for $\cot \psi$ its value as given in (2) in order to arrive at the indicated formula (4).

Corollary 2. The angle ψ between the radius vector of a point P on the curve $\rho = f(\theta)$ and the tangent to the curve at this point is given by the formula

$$\tan \psi = \frac{\rho}{\frac{d\rho}{d\theta}}, \tag{5}$$

it being understood that ρ and $\frac{d\rho}{d\theta}$ are to be evaluated at the given point P This result follows at once by taking the reciprocal of each member of (2), recalling that the reciprocal of $\cot \psi$ is $\tan \psi$.

Example. In the spiral of Archimedes, the polar equation of which is $\rho = a\theta$ where a is a constant (see Fig. page 368), show that the polar subnormal is the same at all points. Investigate also the value of the polar subtangent.

SOLUTION. Since

$$\rho = a\theta,$$

we have

$$\frac{d\rho}{d\theta} = a$$

and therefore, by the foregoing theorem and corollary, we have

$$\textit{polar subnormal} = a = \textit{a constant};$$

$$\textit{polar subtangent} = \frac{\rho^2}{a}.$$

102. Differentials in polar coördinates. If $\rho = f(\theta)$, then, in accordance with the definitions of Art. 98, we have $d\theta = \Delta\theta$ and

$$d\rho = f'(\theta)\,\Delta\theta, \quad \text{or} \quad d\rho = f'(\theta)\,d\theta. \tag{1}$$

A geometrical representation of $d\rho$ may be obtained as follows: Upon the tangent at P to the curve $\rho = f(\theta)$ determine the point T whose perpendicular distance TS from OP (prolonged) is equal to $\rho\Delta\theta$; that is, equal to the circular arc PR' Then $d\rho = PS$, for in the triangle PST we have

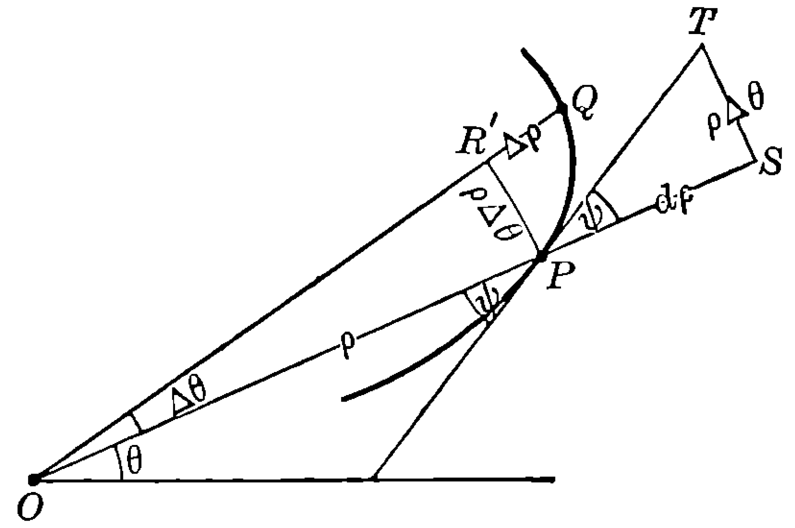

FIG. 113

$$\frac{PS}{TS} = \frac{PS}{\rho\Delta\theta} = \cot\psi,$$

and hence by (2)

$$\frac{PS}{\rho\Delta\theta} = \frac{1}{\rho}\frac{d\rho}{d\theta} = \frac{1}{\rho}f'(\theta).$$

Therefore

$$PS = f'(\theta)\,\Delta\theta. \tag{2}$$

Comparing (2) with (1) we have, as stated, $PS = d\rho$.

The figure also shows $\Delta\rho$, its value being $R'Q$.

Just as it was shown in Art. 99 that dy and Δy differ from each other by an infinitesimal of higher order than either as Δx approaches zero, so we may now show that the infinitesimals $d\rho$ and $\Delta\rho$ have a like property as $\Delta\theta$ approaches zero. The proof is left to the student.

EXERCISES *

1. Prove that the product of the polar subnormal and the polar subtangent of any point on a curve is equal to the square of the radius vector of the point.

2. Find the values of the polar subnormal and the polar subtangent at the point on the circle $\rho = 2 \sin \theta$ for which $\theta = 60°$. Do the same for the point at which $\theta = 120°$ Illustrate the geometric meaning in each case by a suitable drawing.

3. Work Ex. 2, using the circle $\rho = 2 \cos \theta$ instead of $\rho = 2 \sin \theta$, other parts of the data remaining the same. Again make suitable drawings.

4. Prove that at every point of the parabola $\rho = a \sec^2 \dfrac{\theta}{2}$ we have $\theta + 2\psi = \pi$.

5. Prove that at every point of the curve $\rho = a \sin^3 \dfrac{\theta}{3}$ we have $\theta = 3\psi$.

6. Show that ψ is a constant in the logarithmic spiral, $\rho = e^{a\theta}$. Since the tangent thus makes a constant angle with the radius vector, this curve is also known as the equiangular spiral.

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* The student is referred to pages 368–370 for the graphs of various curves in polar coördinates.

7. Prove that at every point of the cardioid $\rho = a(1 - \cos\theta)$ we have $\theta = 2\psi$.

8. Prove that the spiral of Archimedes $\rho = a\theta$ and the reciprocal spiral $\rho = \frac{a}{\theta}$ intersect at right angles.

HINT. Use Corollary 2 of Art. 101.

9. Prove that the two cardioids $\rho = a(1 + \cos\theta)$, $\rho = b(1 - \cos\theta)$ intersect at right angles. Draw a figure.

10. Prove that the reciprocal spiral $\rho = \frac{a}{\theta}$ has a constant polar subtangent.

11. Prove the statement at the end of Art. 102; namely, that $d\rho$ and $\Delta\rho$ differ from each other by an infinitesimal of higher order than either.

12. State the geometric significance of a negative polar subtangent. See footnote, page 203.

13. Show that in the curve $\rho = e^{\theta}$ the polar subtangent is equal to the polar subnormal at every point. Draw a diagram which gives (approximately) the graph of this curve and which shows the geometric meaning of the result just stated.

14. Show that as $\Delta\theta$ approaches zero the length PT in Fig. 113 is an infinitesimal of the same order as either PS or TS, it being assumed that P is a point at which ψ is equal neither to zero nor π.

103. The differential of a function of several variables. Suppose that $z = f(x, y)$ is a function of the two independent variables x and y. Let x and y be regarded for the moment as fixed in value and then allow x to take an increment Δx and y to take an increment Δy. The value of the function will thus become $z + \Delta z = f(x + \Delta x, y + \Delta y)$, and therefore the increment Δz which the function takes is as follows:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{1}$$

As Δx and Δy approach zero, Δz is clearly an infinitesimal. Furthermore, it follows from the theorem of Art. 92 (by placing herein $h = \Delta x$, $k = \Delta y$ and transposing $f(x, y)$ to the left) that Δz, when thus regarded as an infinitesimal, differs from the expression $\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$ by an infinitesimal which is of higher degree than the first in Δx and Δy. Following an obvious analogy with Arts. 98, 99, we therefore define the differential dz of the function $z = f(x, y)$ as follows:

$$dz = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.^* \tag{2}$$

Moreover, it is customary to define $dx = \Delta x$ and $dy = \Delta y$ so that (2) may be written

---

* It may be observed that this expression occurs also as the basis of the approximation formula (2) of Art. 93.

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{3}$$

The differential dz as thus defined is usually called the **total differential** of z.

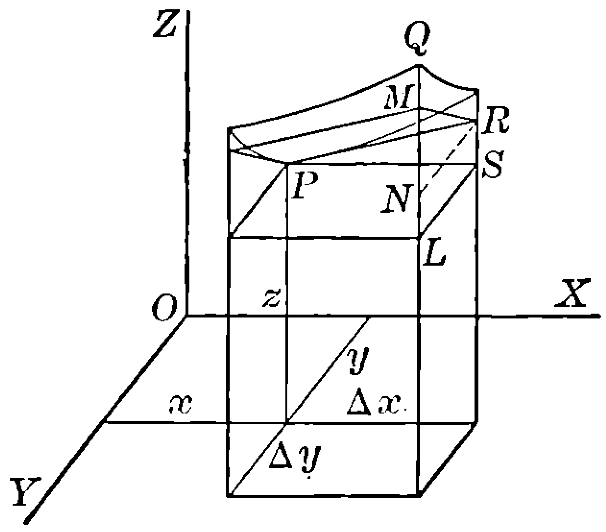
Thus, if $z = x^3 + y^3 - 3\,xy$, we have

$$\frac{\partial z}{\partial x} = 3(x^2 - y), \qquad \frac{\partial z}{\partial y} = 3(y^2 - x),$$

and therefore

$$dz = 3(x^2 - y)dx + 3(y^2 - x)dy.$$

Considered geometrically, Δz is represented by LQ in the accompanying figure (which is a duplicate of Fig. 83), while dz is represented by LM, the latter fact resulting from equation (4) of Art. 63. It is understood, of course, that (as in Fig. 83) PRM represents the tangent plane to the surface $z = f(x, y)$ (represented by PQ) at the given point $P = (x, y, z)$.



The total differential dw of a function $w = f(x, y, z)$ of the three independent variables x, y, z is defined analogously as follows:

$$dw = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.$$

Reasoning from formula (13) of Art. 92, it appears directly that the infinitesimal dw as thus defined differs from Δw by an infinitesimal which is of higher degree than the first in Δx, Δy, and Δz.

EXERCISES

Find the total differential of each of the following functions:

1. $z = x^3 + x^2y^2 + y^3$.

2. $z = x^2 - 2\,xy + y^2 + x - y + 1$.

3. $z = \sin(x + y)$.

4. $z = \tan(x + y)$.

5. $z = \log(x + y)$.

6. $z = \dfrac{x + y}{x - y}$.

7. $z = \cos^2(x - y)$.

8. $r = \sqrt{\theta + \phi}$.

9. $r = \arctan \dfrac{\theta}{\phi}$.

10. $z = e^{x+y}$.

11. $w = e^{x+y+z}$.

12. $w = \sqrt{x^2 + y^2 + z^2}$.

13. $s = \dfrac{t}{\sqrt{t^2 + u^2}}$.

14. $w = \cos(x + y + z)$.

MISCELLANEOUS EXERCISES

1. Find the coördinates of the point P on the curve

$$x = t^2 - 5, \quad y = 2t - 2$$

which is nearest the origin and show that the normal to the curve at this point passes through the origin.

2. At what angles do the curves $y^2 = 4ax$ and $(x^2 + y^2)^2 = 16axy^2$ intersect?
Ans. 0°, 90°, 63° 26′.

3. The tangent at any point P on the curve $x^2y^3 = 32$ meets the X-axis at A and the Y-axis at B. Prove that the ratio $\frac{BP}{PA}$ has a constant value.

4. A spherical ball of radius a is slowly lowered into a cylindrical drinking glass of radius b which is partly filled with water. If the rate at which the ball is lowered is the constant r, find the formula for the rate at which the water level is rising in the glass when the ball is just half immersed. *Ans.* a^2r/b^2.

5. Find the point on the X-axis at which the line segment joining the points (1, 2) and (5, 10) subtends the maximum angle.

6. A small bead moves with constant speed around the circumference of an ellipse. Show that in order for its centrifugal force when at the extremity of the major axis to be twice what it is at the extremity of the minor axis the semiaxes a, b of the ellipse must bear the relation $2b^3 = a^3$.

7. The normal at any point P on a parabola is extended until it meets the directrix in the point D. Prove that the radius of curvature at P is equal to $2DP$.

8. Show that the tangents to the curve

$$y = x^5 - 5x^3 + 5x$$

which are parallel to the X-axis are double tangents.

9. A train is traveling at the rate of 45 miles an hour along a straight track. Find its angular velocity about a point distant 2 miles from the track four minutes after the instant when the train is nearest to that point.

10. Trace the curve

$$x = 2\cos 2\phi, \quad y = 2\cos 3\phi.$$

Show that the curve is a cubic and examine it for the presence of singular points, points of inflection and asymptotes.

11. A point moves from the origin in such a way that the x- and y-components of its velocity after t seconds are respectively $a\cos t$ and $a\sin t$. Prove that its acceleration has a constant magnitude and find the equation of the path.

12. A point P moves in any manner along a straight line in such a way that at the time t its distance from the starting point is s and its velocity is v. Regarding the s-values as abscissas and the v-values as ordinates, suppose that the various points (s, v) corresponding to the motion are now plotted in a coördinate plane, thus leading to a determinate curve in this plane. Show that the acceleration of P in its motion at any point is measured by the length of the subnormal associated with this point on the curve thus drawn.

INTEGRAL CALCULUS

CHAPTER XII

INTEGRATION

I. General Introduction

104. Definitions. In the earlier chapters we have seen how to find the derivative of a function and we have noted various applications of the derivative to geometry, physics, etc. Many important applications of the calculus, however, involve not the finding of the derivative but the inverse process; that is, the derivative of a function is given and it is required to find the function itself.

Thus, if it be required of a function that its derivative shall be x^2, the function may be taken as $\frac{x^3}{3}$ since we have

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{1}{3}\frac{d(x^3)}{dx} = \frac{1}{3}\cdot 3\,x^2 = x^2. \tag{1}$$

Similarly, if the derivative is to be $\sin x$, the function may be taken as $-\cos x$ since we have

$$\frac{d}{dx}(-\cos x) = -\frac{d}{dx}(\cos x) = -(-\sin x) = \sin x. \tag{2}$$

The general problem of determining a function $F(x)$ such that

$$\frac{d}{dx}F(x) = f(x) \quad (\text{or } dF(x) = f(x)dx), \tag{3}$$

where $f(x)$ is a given function of x, is called **integration.** A function $F(x)$ satisfying (3) is called an **integral** of $f(x)$ and is represented by the symbol

$$\int f(x)dx.$$

For example, in the language of integration, equations (1) and (2) become respectively

$$\int x^2\,dx = \frac{x^3}{3}, \qquad \int \sin x\,dx = -\cos x.$$

Similarly, the student may verify at once the correctness of the following:

$$\int x^3\,dx = \frac{x^4}{4}, \qquad \int \cos x\,dx = \sin x, \qquad \int \frac{dx}{1+x^2} = \text{arc}\tan x.$$

$$\int \frac{dx}{x} = \log x, \qquad \int e^x\,dx = e^x, \qquad \int \cot x\,dx = \log \sin x$$

$$\int \sec^2 x\,dx = \tan x, \qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \text{arc}\sin\frac{x}{a}.$$

Historically, the symbol $\int$ is a distorted letter S standing for the word "sum." There is, in fact, an intimate relation between integration and summation, as will be explained in Chapters XV, XVI.

105. Constant of integration. If $F(x)$ is an integral of the function $f(x)$, that is, if $F(x)$ is such that

$$\frac{dF(x)}{dx} = f(x),$$

then the expression $F(x) + C$, where C is any given constant, is also an integral of $f(x)$ since we may write

$$\frac{d}{dx}\,(F(x) + C) = \frac{dF(x)}{dx} + \frac{dC}{dx} = \frac{dF(x)}{dx} + 0 = f(x).$$

In other words, any constant may be added to an integral of $f(x)$ and the result will continue to be an integral. Because of this indeterminateness, the symbol

$$\int f(x)\,dx \tag{1}$$

is called the **indefinite integral** of $f(x)$, and it is customary to replace it by the symbol

$$\int f(x)dx + C$$

whenever this indeterminateness is to be stressed, the letter C standing for a constant whose value is altogether arbitrary. As thus employed, C is called the **constant of integration.**

For example, instead of

$$\int x^2\,dx = \frac{x^3}{3}$$

we write

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

(Compare Art. 104.)

Likewise, instead of $\int \sin x\,dx = -\cos x$ we write $\int \sin x\,dx = -\cos x + C$.

Similarly, $\int x^3\,dx = \frac{x^4}{4} + C$, $\int \cos x\,dx = \sin x + C$,

$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \frac{dx}{x} = \log x + C.$$

106. Geometric significance of integration. The geometric significance of integration is readily understood from an example:

Example. Determine the equation of the curve at every point of which the tangent has a slope equal to twice the abscissa of the point.

SOLUTION. By Art. 9 the question is seen to be that of finding a function y such that

$$\frac{dy}{dx} = 2\,x. \tag{1}$$

In other words, the problem is that of integrating $2\,x$. This is readily done by inspection. Thus

$$y = \int 2\,x\,dx + C = x^2 + C, \tag{2}$$

where C is the (arbitrary) constant of integration. Therefore, the desired curve may be taken as any one of the family of curves obtained by giving different values to the constant C in (2). All such curves are evidently parabolas situated as indicated in Fig. 114.

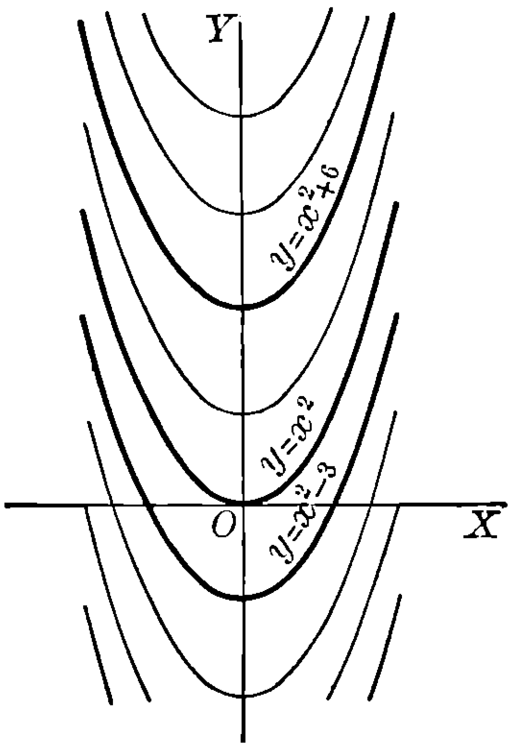


FIG. 114

In interpreting Fig. 114 it is to be observed that the slope of each parabola is the same for any given abscissa x, as follows from (1). Hence, if any line be drawn parallel to the Y-axis and the points P_1, P_2, P_3, at which it meets the various curves be marked, then the tangents drawn to the parabolas at these points will all have the same slope; that is, they will be parallel. Moreover, since the y of any one curve differs from the y of any other only by some constant amount, as follows from (2), it appears that all the parabolas of the family can be obtained by merely moving any one of them vertically up or down.

Analogous geometrical statements apply to any case of integration.

EXERCISES

1. Reasoning from the definition of an integral, confirm the correctness of each of the following equations:

(*a*) $\int 3x^2\,dx = x^3 + C$;

(*b*) $\int (x+3)\,dx = \frac{x^2}{2} + 3x + C$;

(*c*) $\int \frac{dx}{x^2} = -\frac{1}{x} + C$;

(*d*) $\int \sec^2 x\,dx = \tan x + C$;

(*e*) $\int e^x\,dx = e^x + C$;

(*f*) $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$.

2. Supply by inspection the values of each of the following integrals, and check your answer by differentiation:

(*a*) $\int (2x+1)\,dx$;

(*b*) $\int (1+x)^2\,dx$;

(*c*) $\int \frac{dx}{x+1}$;

(*d*) $\int e^{2x}\,dx$;

(*e*) $\int \frac{dx}{\sqrt{1-x^2}}$;

(*f*) $\int \sin 2x\,dx$.

3. Draw diagrams analogous to Fig. 114 to show the geometric significance of the results in Ex. 1 (*a*), (*b*), (*c*).

107. Nature of integration. It is a familiar fact in arithmetic that if one integer be multiplied by another the product is itself an integer, but it is an equally familiar fact that a like statement cannot be made with reference to the division of one integer by another. For example, in dividing 4 by 3 the quotient is not itself an integer. Otherwise stated, the process of multiplication can be performed at pleasure upon the positive integers without leaving that system of numbers, whereas the inverse process, namely, that of division, cannot be performed in all cases without entering into a further number system, namely, that of the rational fractions. In a similar manner the process of differentiation, when performed upon any one of the elementary functions * leads, as we have seen in Chap. III, to a resulting function which belongs likewise to those classed as elementary, but we are not to expect that the inverse process, namely, the integration of a function, will preserve this simple property. However, it is a fact that a large number of very important integrals *are* expressible in terms of the elementary functions, and it will be the purpose of the remainder of this chapter to make a study of such integrals, learning how first to identify them and subsequently to evaluate them.

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* The elementary functions are generally taken to include (1) all such as result from purely algebraic processes (addition, subtraction, multiplication, division, involution, evolution); (2) the trigonometric functions (sine, cosine, etc.) and their inverses (arc sine, arc cosine, etc.); (3) the exponential function a^x and its inverse $\log_a x$.

108. Standard integral forms. It is desirable before attempting the actual technique of integration to tabulate for future reference the following twenty-three standard integral forms. The first thirteen of these are but the equivalents of certain standard formulas already appearing in Chap. III, the only difference being that, as stated below, these formulas are in the language of integration rather than that of differentiation. The remaining forms, namely, (14)–(23), are readily established as will appear presently.

The list is not intended for memorizing as a whole, at least at this stage, but the student should gradually fix each in mind as he proceeds and sees for himself its full import. It is to be understood that u and v represent throughout any given functions of some independent variable, as x.

STANDARD INTEGRAL FORMS

$$\int (du + dv) = \int du + \int dv. \qquad [1]$$

$$\int a\,du = a\int du. \qquad [2]$$

$$\int dx = x + C. \qquad [3]$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C. \quad (n \neq -1) \qquad [4]$$

$$\int \frac{du}{u} = \log u + C. \qquad [5]$$

$$\int a^u\,du = \frac{a^u}{\log a} + C. \qquad [6]$$

$$\int e^u\,du = e^u + C. \qquad [7]$$

$$\int \sin u\,du = -\cos u + C. \qquad [8]$$

$$\int \cos u\,du = \sin u + C. \qquad [9]$$

$$\int \sec^2 u\,du = \tan u + C. \qquad [10]$$

$$\int \csc^2 u\,du = -\cot u + C. \qquad [11]$$

$$\int \sec u \tan u\,du = \sec u + C. \qquad [12]$$

$$\int \csc u \cot u\,du = -\csc u + C. \qquad [13]$$

$$\int \tan u \, du = \log \sec u + C. \qquad [14]$$

$$\int \cot u \, du = \log \sin u + C. \qquad [15]$$

$$\int \sec u \, du = \log (\sec u + \tan u) + C. \qquad [16]$$

$$\int \csc u \, du = \log (\csc u - \cot u) + C. \qquad [17]$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \operatorname{arc} \tan \frac{u}{a} + C. \qquad [18]$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u - a}{u + a} + C. \qquad [19]$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \operatorname{arc} \sin \frac{u}{a} + C. \qquad [20]$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2}) + C. \qquad [21]$$

$$\int \frac{du}{\sqrt{2au - u^2}} = \operatorname{arc} \operatorname{vers} \frac{u}{a} + C. \qquad [22]$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arc} \sec \frac{u}{a} + C. \qquad [23]$$

109. Integral forms [1]–[5]. The standard integral form [1] is but another form for the derivative formula occurring in Theorem I of Art. 15. It may be stated in words as follows:

The integral of the sum of two functions is equal to the sum of their integrals.

Illustration:

$$\int (x + x^2)\, dx = \int x\, dx + \int x^2\, dx = \frac{x^2}{2} + \frac{x^3}{3} + C.$$

Standard form [2] is equivalent to the third formula in Art. 15, and may be stated as follows:

The integral of a constant multiplied by a function is equal to the constant multiplied by the integral of the function. In other words, a constant factor may always be taken outside the integral sign.

Illustration:

$$\int 3x\, dx = 3\int x\, dx = 3 \cdot \frac{x^2}{2} + C = \frac{3}{2} x^2 + C.$$

Standard form [3] evidently is the counterpart of the simple equation $\frac{dx}{dx} = 1$, while forms [4] and [5] express respectively the formulas of Arts. 17 and 24 pertaining to the differential calculus.

Illustration:

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C;\quad \int \frac{dx}{x+1} = \int \frac{d(x+1)}{x+1} = \log(x+1) + C.$$

EXERCISES

Establish each of the following results, and check each by means of differentiation:

1. $\int x^3\,dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C.$

2. $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + C.$

3. $\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} + C = -x^{-1} + C = -\frac{1}{x} + C.$

4. $\int ax^6\,dx = a\int x^6\,dx = \frac{ax^7}{7} + C.$

5. $\int x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}} + C.$

6. $\int x^{\frac{2}{3}}\,dx = \frac{3}{5}x^{\frac{5}{3}} + C.$

7. $\int t^{\frac{3}{2}}dt = \frac{2}{5}t^{\frac{5}{2}} + C.$

8. $\int \frac{2\,dx}{x^3} = -\frac{1}{x^2} + C.$

9. $\int \sqrt{2\,px}\,dx = \frac{2}{3}x\sqrt{2\,px} + C.$

10. $\int \frac{2\,ds}{\sqrt[3]{s}} = 3\,s^{\frac{2}{3}} + C.$

11. $\int 3\,a\theta^2\,d\theta = a\theta^3 + C.$

12. $\frac{5}{4}\int \sqrt[4]{z}\,dz = z^{\frac{5}{4}} + C.$

13. $\frac{1}{\sqrt[n]{n}}\int (nx)^{\frac{1-n}{n}}\,dx = x^{\frac{1}{n}} + C.$

14. $2\int \frac{dx}{x} = \log x^2 + C.$

15. $\int (3\,x^3 - 4\,x^2 + 2\,x - 2)\,dx.$

Solution.

$$\begin{aligned}\int (3\,x^3 - 4\,x^2 + 2\,x - 2)\,dx &= \int 3\,x^3\,dx - \int 4\,x^2\,dx + \int 2\,x\,dx - \int 2\,dx &&\text{by [1]}\\ &= 3\int x^3\,dx - 4\int x^2\,dx + 2\int x\,dx - 2\int dx &&\text{by [2]}\\ &= \tfrac{3}{4}x^4 - \tfrac{4}{3}x^3 + x^2 - 2\,x + C. &&\text{by [4]}\end{aligned}$$

16. $\int (ax^4 + bx^3 + cx^2 + dx + e)\,dx = \frac{ax^5}{5} + \frac{bx^4}{4} + \frac{cx^3}{3} + \frac{dx^2}{2} + ex + C.$

17. $\int \left(\sqrt[3]{x^2} - \frac{1}{\sqrt[3]{x^2}} + \frac{2}{x^3}\right)dx = \frac{3}{5}x^{\frac{5}{3}} - 3\,x^{\frac{1}{3}} - \frac{1}{x^2} + C.$

18. $\int(\sqrt{a}+\sqrt{x})^2\,dx = ax + \frac{4\sqrt{a}}{3}\sqrt{x^3} + \frac{x^2}{2} + C.$

HINT. First expand $(\sqrt{a}+\sqrt{x})^2$ by the binomial theorem.

19. $\int(a^{\frac{2}{3}} - x^{\frac{2}{3}})^3\,dx = a^2x + \frac{9}{7}a^{\frac{2}{3}}x^{\frac{7}{3}} - \frac{9}{5}a^{\frac{4}{3}}x^{\frac{5}{3}} - \frac{x^3}{3} + C.$

20. $\int\frac{1}{2}(a-x)^3\sqrt{x}\,dx = \frac{a^3x^{\frac{3}{2}}}{3} - \frac{3\,a^2x^{\frac{5}{2}}}{5} + \frac{3\,ax^{\frac{7}{2}}}{7} - \frac{x^{\frac{9}{2}}}{9} + C.$

21. $\int(a-x)(b+x)\,dx = abx + \frac{1}{2}(a-b)\,x^2 - \frac{x^3}{3} + C.$

22. $\int(1+x)^{\frac{1}{2}}\,dx = \frac{2}{3}(1+x)^{\frac{3}{2}} + C.$

SOLUTION. This may be brought under [4] as follows: First, recalling the meaning of dx (see Art. 98) we may write, if desired, $dx = d(1+x)$. Hence, for the integral in question we may write

$$\int(1+x)^{\frac{1}{2}}\,dx = \int(1+x)^{\frac{1}{2}}\,d(1+x).$$

As now written, the integral falls under [4] with $u = 1 + x$ and $n = \frac{1}{2}$. Thus,

$$\int(1+x)^{\frac{1}{2}}\,d(1+x) = \frac{(1+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}(1+x)^{\frac{3}{2}} + C.$$

23. $\int(a^2+b^2x^2)^{\frac{1}{2}}\,x\,dx = \frac{1}{3\,b^2}(a^2+b^2x^2)^{\frac{3}{2}} + C.$

HINT. This may be brought under [4] as follows: First observe that $d(a^2+b^2x^2) = 2\,b^2x\,dx$. It is therefore natural to multiply under the integral sign by the constant $2\,b^2$ and divide by the same quantity outside the integral sign, thus leaving the value of the integral unchanged but giving the whole the form

$$\frac{1}{2\,b^2}\int(a^2+b^2x^2)^{\frac{1}{2}}\,2\,b^2x\,dx = \frac{1}{2\,b^2}\int(a^2+b^2x^2)^{\frac{1}{2}}\,d(a^2+b^2x^2).$$

The last form is evidently [4] with $u = a^2 + b^2x^2$ and $n = \frac{1}{2}$.

24. $\int(a+bx)^{\frac{1}{3}}\,dx = \frac{3}{4\,b}(a+bx)^{\frac{4}{3}} + C.$

25. $\int\sqrt{a^2-x^2}\,x\,dx = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C.$

26. $\int(a^3+x^3)^{\frac{1}{3}}\,x^2\,dx = \frac{1}{4}(a^3+x^3)^{\frac{4}{3}} + C.$

27. $\int\frac{x^2\,dx}{\sqrt{a^3+x^3}} = \frac{2}{3}\sqrt{a^3+x^3} + C.$

28. $\int\frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C.$

29. $\int\sqrt{3\,x^2+2\,x+1}\,(3\,x+1)\,dx = \frac{1}{3}(3\,x^2+2\,x+1)^{\frac{3}{2}} + C.$

HINT. Use [4] with $u = 3\,x^2 + 2\,x + 1$, hence $du = (6\,x+2)\,dx$.

30. $\displaystyle\int \frac{(x^2 + x + 1)\,dx}{(2\,x^3 + 3\,x^2 + 6\,x + 1)^{\frac{1}{4}}} = \frac{2}{9}(2\,x^3 + 3\,x^2 + 6\,x + 1)^{\frac{3}{4}} + C.$

31. $\displaystyle\int (1 + e^x)^{\frac{1}{2}} e^x\,dx = \tfrac{2}{3}(1 + e^x)^{\frac{3}{2}} + C.$

32. $\displaystyle\int \sin^2 x \cos x\,dx = \int (\sin x)^2 \cos x\,dx = \frac{(\sin x)^3}{3} + C = \frac{\sin^3 x}{3} + C.$

33. $\displaystyle\int \sin^3 ax \cos ax\,dx = \frac{1}{4\,a}\sin^4 ax + C.$

34. $\displaystyle\int \cos^2 x \sin x\,dx = -\int \cos^2 x\,(-\sin x\,dx) = -\frac{\cos^3 x}{3} + C.$

35. $\displaystyle\int \sqrt{\sin x}\cos x\,dx = \tfrac{2}{3}\sqrt{\sin^3 x} + C.$

36. $\displaystyle\int \tan^2 ax \sec^2 ax\,dx = \frac{1}{3\,a}\tan^3 ax + C.$

37. $\displaystyle\int \frac{(\log x)^2}{x}\,dx = \frac{1}{3}(\log x)^3 + C.$

38. $\displaystyle\int \frac{(\log ax)^m}{x}\,dx = \frac{1}{m+1}(\log ax)^{m+1} + C.$

39. $\displaystyle\int \frac{dx}{ax + b} = \frac{1}{a}\log(ax + b) + C.$

HINT. We may write

$$\int \frac{dx}{ax + b} = \frac{1}{a}\int \frac{d(ax + b)}{ax + b}.\quad \text{Now apply [5].}$$

40. $\displaystyle\int \frac{x\,dx}{x^2 + 1} = \frac{1}{2}\log(x^2 + 1) + C = \log(x^2 + 1)^{\frac{1}{2}} + C.$

41. $\displaystyle\int \frac{x^2\,dx}{x^3 - 1} = \log(x^3 - 1)^{\frac{1}{3}} + C.$

42. $\displaystyle\int \frac{\sin x\,dx}{a + b\cos x} = -\frac{1}{b}\log(a + b\cos x) + C.$

43. $\displaystyle\int \frac{\sec^2 x\,dx}{a + b\tan x} = \frac{1}{b}\log(a + b\tan x) + C.$

44. $\displaystyle\int \frac{e^{ax}\,dx}{e^{ax} + b} = \frac{1}{a}\log(e^{ax} + b) + C.$

45. $\displaystyle\int \frac{x^2\,dx}{x + 1} = \frac{x^2}{2} - x + \log(x + 1) + C.$

HINT. First divide numerator by denominator, obtaining quotient and remainder.

46. $\displaystyle\int \frac{x^3\,dx}{x - 1} = \frac{x^3}{3} + \frac{x^2}{2} + x + \log(x - 1) + C.$

47. $\displaystyle\int \frac{ax + b}{cx + d}\,dx = \frac{a}{c}x + \frac{bc - ad}{c^2}\log(cx + d) + C.$

48. $\displaystyle\int \frac{y^2 + 1}{y - 1}\,dy = \frac{y^2}{2} + y + 2\log(y - 1) + C.$

49. $\displaystyle\int \frac{e^{3s}ds}{e^s - 1} = \frac{e^{2s}}{2} + e^s + \log(e^s - 1) + C.$

50. $\displaystyle\int \frac{(\log x)^2\,dx}{x(1 - \log x)} = -\frac{(\log x)^2}{2} - \log x + \log(1 - \log x) + C.$

51. Find each of the following integrals and verify each result by differentiation:

(a) $\displaystyle\int \sqrt{3x+1}\,dx;$

(b) $\displaystyle\int (\sqrt{x} + \sqrt{x+1})^2(2x+1)\,dx;$

(c) $\displaystyle\int \frac{x^2\,dx}{\sqrt[3]{a^3 - x^3}};$

(d) $\displaystyle\int \frac{(a^{\frac{1}{3}} - z^{\frac{1}{3}})^2}{\sqrt{2}}\,dz;$

(e) $\displaystyle\int \frac{x^3+2}{x+2}\,dx;$

(f) $\displaystyle\int \frac{(x-1)\,dx}{x^2 - 2x + 3};$

(g) $\displaystyle\int \sin mx \cos mx\,dx;$

(h) $\displaystyle\int \sin^m x \cos x\,dx;\ m \neq -1.$

(i) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = ?$

(j) $\displaystyle\int \frac{\csc^2\phi\,d\phi}{a - b\cot\phi};$

(k) $\displaystyle\int \frac{e^r - 1}{e^r + 1}\,dr;$

(l) $\displaystyle\int \frac{1 + \cos^2 x}{1 + \cos x}\sin x\,dx.$

110. Integral forms [6]–[17]. Standard forms [6] and [7] are but other ways of expressing the two formulas of Art. 25, while forms [8]–[13] correspond in a similar sense to the derivative formulas to be found in Arts. 19–23.

As to forms [14]–[17], these require proof which may be supplied as follows:

PROOF OF [14].

$$\int \tan u\,du = \int \frac{\sin u\,du}{\cos u} = -\int \frac{-\sin u\,du}{\cos u} = -\int \frac{d(\cos u)}{\cos u}$$
$$= -\log\cos u + C$$
$$= -\log\frac{1}{\sec u} + C = -[\log 1 - \log\sec u] + C$$
$$= \log\sec u + C. \qquad \text{By [5]}$$

PROOF OF [15].

$$\int \cot u\,du = \int \frac{\cos u\,du}{\sin u} = \int \frac{d(\sin u)}{\sin u} = \log\sin u + C. \qquad \text{By [5]}$$

PROOF OF [16]. Since

$$\sec u = \sec u\,\frac{\sec u + \tan u}{\sec u + \tan u} = \frac{\sec u\tan u + \sec^2 u}{\sec u + \tan u},$$

$$\int \sec u\,du = \int \frac{\sec u\tan u + \sec^2 u}{\sec u + \tan u}\,du = \int \frac{d(\sec u + \tan u)}{\sec u + \tan u}$$
$$= \log(\sec u + \tan u) + C. \qquad \text{By [5]}$$

PROOF OF [17]. Since

$$\csc u = \csc u \frac{\csc u - \cot u}{\csc u - \cot u} = \frac{-\csc u \cot u + \csc^2 u}{\csc u - \cot u},$$

$$\int \csc u\, du = \int \frac{-\csc u \cot u + \csc^2 u}{\csc u - \cot u} du = \int \frac{d(\csc u - \cot u)}{\csc u - \cot u}$$

$$= \log(\csc u - \cot u) + C. \qquad \text{By [5]}$$

EXERCISES

Establish each of the following results, and check each by means of differentiation:

1. $\int 4\, a^{3x}\, dx = \frac{4\, a^{3x}}{3 \log a} + C.$

SOLUTION.

$$\int 4\, a^{3x}\, dx = 4 \int a^{3x}\, dx = \tfrac{4}{3} \int a^{3x}\, 3\, dx = \tfrac{4}{3} \int a^{3x} d(3\, x) \qquad \text{By [2]}$$

$$= \frac{4}{3} \frac{a^{3x}}{\log a} + C = \frac{4\, a^{3x}}{3 \log a} + C. \qquad \text{By [6]}$$

2. $\int 2\, e^{4x}\, dx = \tfrac{1}{2}\, e^{4x} + C.$

3. $\int a^{2x-1}\, dx = \frac{a^{2x-1}}{2 \log a} + C.$

4. $\int 5^{x^2}\, x\, dx = \frac{5^{x^2}}{2 \log 5} + C.$

5. $\int a^x e^x\, dx = \frac{a^x e^x}{1 + \log a} + C.$

6. $\int a e^{-mx}\, dx = -\frac{a}{m} e^{-mx} + C.$

7. $\int (e^{mx})^2\, dx = \frac{e^{2mx}}{2\, m} + C.$

8. $\int \frac{dt}{\sqrt{e^t}} = -\frac{2}{\sqrt{e^t}} + C.$

9. $\int a^{\log x} \frac{dx}{x} = \frac{a^{\log x}}{\log a} + C.$

10. $\int (e^x + e^{-x})\, dx = e^x - e^{-x} + C.$

11. $\int e^{x^2+4x+2}(x + 2)\, dx = \tfrac{1}{2}\, e^{x^2+4x+2} + C.$

12. $\int (a^{nx} - b^{mx})\, dx = \frac{a^{nx}}{n \log a} - \frac{b^{mx}}{m \log b} + C.$

13. $\int (e^x - e^{-x})^2\, dx = \tfrac{1}{2}(e^{2x} - e^{-2x}) - 2\, x + C.$

14. $\int \frac{(a^x - b^x)^2}{a^x b^x}\, dx = \frac{a^x b^{-x} - a^{-x} b^x}{\log a - \log b} - 2\, x + C.$

15. $\int (e^t + e^{-t})^3\, dt = \frac{e^{3t}}{3} + 3\, e^t - 3\, e^{-t} - \frac{e^{-3t}}{3} + C.$

16. $\int e^{\sin x} \cos x\, dx = e^{\sin x} + C.$

17. $\int e^{\cos x} \sin x\, dx = -\, e^{\cos x} + C.$

18. $\int \frac{a^{\tan x}}{\cos^2 x}\, dx = \frac{a^{\tan x}}{\log a} + C.$

19. $\int a^{\log \sin x} \cot x\, dx = \frac{a^{\log \sin x}}{\log a} + C.$

20. $\int \sin 4x\,dx = -\frac{\cos 4x}{4} + C.$

SOLUTION.

$$\int \sin 4x\,dx = \tfrac{1}{4}\int \sin 4x \quad 4\,dx = \tfrac{1}{4}\int \sin 4x \quad d(4x) \qquad \text{By [2]}$$
$$= \frac{1}{4}(-\cos 4x) + C = -\frac{\cos 4x}{4} + C. \qquad \text{By [8]}$$

21. $\int \cos 5x\,dx = \tfrac{1}{5}\sin 5x + C.$

22. $\int \sec^2 mx\,dx = \frac{1}{m}\tan mx + C.$

23. $\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax + C.$

24. $\int \sec at \tan at\,dt = \frac{1}{a}\sec at + C.$

25. $\int \frac{\csc ay}{\tan ay}\,dy = -\frac{1}{a}\csc ay + C.$

26. $\int \tan bx\,dx = \frac{1}{b}\log \sec bx + C.$

27. $\int \cot \frac{x}{a}\,dx = a \log \sin \frac{x}{a} + C.$

28. $\int \sec \frac{x}{a}\,dx = a \log\left(\sec \frac{x}{a} + \tan \frac{x}{a}\right) + C.$

29. $\int \sec^2 x^3 \cdot x^2\,dx = \tfrac{1}{3}\tan x^3 + C.$

30. $\int \cos(\log x)\frac{dx}{x} = \sin(\log x) + C.$

31. $\int \frac{1 + \sin x}{\cos x}\,dx = -\log(1 - \sin x) + C.$

HINT. Write $\frac{1 + \sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x.$

32. $\int \frac{a + b\cos mx}{\sin mx}\,dx = \frac{a}{m}\log(1 - \cos mx) + \frac{b - a}{m}\log \sin mx + C.$

33. $\int (\tan y + \cot y)^2\,dy = \tan y - \cot y + C.$

34. $\int (\sec x - \tan x)^2\,dx = 2(\tan x - \sec x) - x + C.$

35. $\int (\tan x + 1)^2\,dx = \tan x + \log \sec^2 x + C.$

36. $\int (\sin x + \cos x)^2\,dx = x + \sin^2 x + C.$

37. $\int \frac{\sin x + \cos x}{\tan x}\,dx = \sin x + \cos x + \log(\csc x - \cot x) + C.$

38. $\int \frac{dx}{1 + \cos x} = -\cot x + \csc x + C.$

HINT. Multiply both numerator and denominator by $1 - \cos x$.

39. $\int \frac{dx}{1 + \sin x} = \tan x - \sec x + C.$

40. $\int \frac{\sec^2 x}{a + b\tan x}dx = \frac{1}{b}\log(a + b\tan x) + C.$

41. Find each of the following integrals and verify each result by differentiation.

(a) $\int 2^{2x}\,dx$;
(b) $\int 3^{t^2} t\,dt$;
(c) $\int \frac{a\,dx}{\sqrt{e^x}}$;
(d) $\int ba^{x^3}x^2\,dx$;
(e) $\int 3^{2\sin x}\cos x\,dx$;
(f) $\int a^{b\cos t}\sin t\,dt$;
(g) $\int e^{x^2-3x}\,dx$;
(h) $\int (a^x + a^{-x})^2\,dx$;
(i) $\int \frac{a^{\log x}}{x}\,dx$;
(j) $\int x\sin(x^2+1)\,dx$;
(k) $\int \frac{dx}{\cos^2 ax}$;
(l) $\int \tan^2(2x+1)\,dx$;
(m) $\int \cot^2(x-1)\,dx$;
(n) $\int (\tan\phi + \sec\phi)^2\,d\phi$;
(o) $\int \frac{\sin(\log x)}{x}\,dx$;
(p) $\int \frac{a + b\sec^2 x}{\tan x}\,dx$;
(q) $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$;
(r) $\int \frac{e^{a+b\sin x}}{\sec x}\,dx$.

111. Integral forms [18]–[23]. The correctness of each of these forms may be established without difficulty by direct proofs similar to those occurring in Art. 110, but for brevity one may establish any one form by merely differentiating its right member and then showing that the result is the expression under the integral sign on the left. Thus, in order to establish [18] we have by the theorem of Art. 30

$$\frac{d}{dx}\left(\frac{1}{a}\arc\tan\frac{u}{a}\right) = \frac{1}{a}\,\frac{\frac{1}{a}\cdot\frac{du}{dx}}{1+\frac{u^2}{a^2}} = \frac{1}{a^2}\,\frac{\frac{du}{dx}}{\frac{a^2+u^2}{a^2}} = \frac{\frac{du}{dx}}{a^2+u^2}.$$

Hence,

$$d\left(\frac{1}{a}\arc\tan\frac{u}{a}\right) = \frac{du}{a^2+u^2},$$

and this equation is seen to be equivalent to the form [18]. We shall assume henceforth that the remaining forms [19]–[23] have been similarly established.

EXERCISES

Perform and verify each of the following integrations:

1. $\int \frac{dx}{9x^2+16} = \frac{1}{12}\arc\tan\frac{3x}{4} + C.$

SOLUTION. This integral may be brought under form [18] in which $u = 3x$ and $a = 4$. Thus, we may write

$$\int \frac{dx}{9x^2+16} = \frac{1}{3}\int \frac{d(3x)}{(3x)^2+4^2} = \frac{1}{3}\cdot\frac{1}{4}\arc\tan\frac{3x}{4} + C = \frac{1}{12}\arc\tan\frac{3x}{4} + C.$$

2. $\int \frac{dx}{4x^2+25} = \frac{1}{10}\arc\tan\frac{2x}{5} + C.$

3. $\int \frac{dx}{9x^2 - 16} = \frac{1}{24} \log \frac{3x - 4}{3x + 4} + C.$

4. $\int \frac{dx}{4x^2 - 25} = \frac{1}{20} \log \frac{2x - 5}{2x + 5} + C.$

5. $\int \frac{dx}{\sqrt{16 - 9x^2}} = \frac{1}{3} \text{arc sin} \frac{3x}{4} + C.$

6. $\int \frac{dx}{\sqrt{9x^2 + 16}} = \frac{1}{3} \log (3x + \sqrt{9x^2 + 16}) + C.$

7. $\int \frac{dx}{\sqrt{9x^2 - 16}} = \frac{1}{3} \log (3x + \sqrt{9x^2 - 16}) + C.$

8. $\int \frac{dx}{3x^2 + 5} = \frac{1}{\sqrt{15}} \text{arc tan} \frac{\sqrt{15}}{5} x + C.$

9. $\int \frac{dx}{3x^2 - 5} = \frac{1}{2\sqrt{15}} \log \frac{\sqrt{3}x - \sqrt{5}}{\sqrt{3}x + \sqrt{5}} + C.$

10. $\int \frac{5x^2\,dx}{5 - x^6} = \frac{\sqrt{5}}{6} \log \frac{x^3 + \sqrt{5}}{x^3 - \sqrt{5}} + C.$

11. $\int \frac{5x^2\,dx}{\sqrt{5 - x^6}} = \frac{5}{3} \text{arc sin} \frac{x^3}{\sqrt{5}} + C.$

12. $\int \frac{dx}{\sqrt{6x - x^2}} = \text{arc vers} \frac{x}{3} + C.$

Hint. Write

$$\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{2 \cdot 3x - x^2}}.$$

Form [22] now applies in which $u = x$ and $a = 3$.

13. $\int \frac{dx}{\sqrt{10x - x^2}} = \text{arc vers} \frac{x}{5} + C.$

14. $\int \frac{dx}{\sqrt{10x - 4x^2}} = \frac{1}{2} \text{arc vers} \frac{4x}{5} + C.$

15. $\int \frac{dx}{x\sqrt{4x^2 - 9}} = \frac{1}{3} \text{arc sec} \frac{2x}{3} + C.$

Hint. Write

$$\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2 - 3^2}} = \int \frac{d(2x)}{2x\sqrt{(2x)^2 - 3^2}}.$$

Form [23] now applies in which $u = 2x$ and $a = 3$.

16. $\int \frac{dx}{x\sqrt{9x^2 - 16}} = \frac{1}{4} \text{arc sec} \frac{3x}{4} + C.$

17. $\int \frac{dx}{x\sqrt{3x^2 - 5}} = \frac{1}{\sqrt{5}} \text{arc sec} \sqrt{\frac{3}{5}}\, x + C.$

18. $\int \frac{dx}{\sqrt{5x - 3x^2}} = \frac{1}{\sqrt{3}} \text{arc vers} \frac{6x}{5} + C.$

19. $\displaystyle\int \frac{dx}{x\sqrt{9x^4 - 5}} = \frac{1}{2\sqrt{5}} \operatorname{arc\,sec} \frac{3x^2}{\sqrt{5}} + C.$

20. $\displaystyle\int \frac{dt}{a^2 - b^2t^2} = \frac{1}{2ab} \log \frac{bt + a}{bt - a} + C.$

21. $\displaystyle\int \frac{x\,dx}{x^4 + a^4} = \frac{1}{2a^2} \operatorname{arc\,tan} \frac{x^2}{a^2} + C.$

22. $\displaystyle\int \frac{\cos x\,dx}{a^2 + \sin^2 x} = \frac{1}{a} \operatorname{arc\,tan} \left(\frac{\sin x}{a}\right) + C.$

23. $\displaystyle\int \frac{dx}{x\sqrt{1 - (\log x)^2}} = \operatorname{arc\,sin} (\log x) + C.$

24. $\displaystyle\int \frac{dx}{x^2 + 4x + 13} = \frac{1}{3} \operatorname{arc\,tan} \frac{x + 2}{3} + C.$

HINT. Write

$$\int \frac{dx}{x^2 + 4x + 13} = \int \frac{dx}{(x^2 + 4x + 4) + 9} = \int \frac{dx}{(x + 2)^2 + 9} = \int \frac{d(x + 2)}{(x + 2)^2 + 3^2}.$$

Form [18] now applies in which $u = x + 2$, $a = 3$.

25. $\displaystyle\int \frac{dx}{x^2 - 4x + 5} = \operatorname{arc\,tan} (x - 2) + C.$

26. $\displaystyle\int \frac{dx}{\sqrt{1 - 2x - 2x^2}} = \frac{1}{\sqrt{2}} \operatorname{arc\,sin} \frac{2x + 1}{\sqrt{3}} + C.$

HINT. Write

$$\int \frac{dx}{\sqrt{1 - 2x - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{1}{2} - (x^2 + x)}} = \frac{1}{\sqrt{2}} \int \frac{d(x + \frac{1}{2})}{\sqrt{\frac{3}{4} - (x + \frac{1}{2})^2}}.$$

Form [20] now applies in which $u = x + \dfrac{1}{2}$, $a = \dfrac{\sqrt{3}}{2}$.

27. $\displaystyle\int \frac{dx}{\sqrt{2 + x - x^2}} = \operatorname{arc\,sin} \frac{2x - 1}{3} + C.$

28. $\displaystyle\int \frac{5\,dx}{3x^2 - 4x + 3} = \sqrt{5} \operatorname{arc\,tan} \frac{3x - 2}{\sqrt{5}} + C.$

29. $\displaystyle\int \frac{dx}{\sqrt{3x - x^2 - 2}} = \operatorname{arc\,sin} (2x - 3) + C.$

30. $\displaystyle\int \frac{dx}{x^2 - 4x - 5} = \frac{1}{6} \log \frac{x - 5}{x + 1} + C.$

31. $\displaystyle 2\int \frac{dx}{4x^2 - 12x + 5} = \frac{1}{4} \log \frac{2x - 5}{2x - 1} + C.$

32. $\displaystyle\int \frac{dx}{3x^2 - 4x} = \frac{1}{4} \log \frac{3x - 4}{3x} + C$

33. $\displaystyle 3\int \frac{dx}{\sqrt{3x^2 - 4x}} = \sqrt{3} \log (3x - 2 + \sqrt{9x^2 - 12x}) + C.$

34. $\displaystyle\int \frac{dx}{\sqrt{2ax + x^2}} = \log (x + a + \sqrt{2ax + x^2}) + C.$

35. $\int \frac{3x^2\,dx}{\sqrt{x^3 - 9x^6}} = \frac{1}{3} \text{arc vers } 18x^3 + C.$

36. $\int \frac{3x - 2}{4x^2 + 1}\,dx = \frac{3}{8}\log(4x^2 + 1) - \text{arc tan } 2x + C.$

HINT. Write

$$\frac{3x - 2}{4x^2 + 1} = \frac{3x}{4x^2 + 1} - \frac{2}{4x^2 + 1}.$$

37. $\int \frac{5x - 1}{2x^2 + 3}\,dx = \frac{5}{4}\log(2x^2 + 3) - \frac{1}{\sqrt{6}} \text{arc tan } \frac{2x}{\sqrt{6}} + C.$

38. $\int \frac{5x - 1}{\sqrt{3 - 2x^2}}\,dx = -\frac{5}{2}\sqrt{3 - 2x^2} - \frac{1}{\sqrt{2}} \text{arc sin } \frac{2x}{\sqrt{6}} + C.$

39. $\int \frac{3x - 2}{\sqrt{4x - x^2}}\,dx = -3\sqrt{4x - x^2} + 4 \text{ arc sin } \frac{x - 2}{2} + C.$

40. $\int \frac{8x^3 + 4x + 13}{4x^2 - 4x + 5}\,dx = x^2 + 2x + \frac{1}{4}\log(4x^2 - 4x + 5) + \text{arc tan}\left(x - \frac{1}{2}\right) + C.$

41. $\int \frac{5x - 1}{\sqrt{3x^2 - 9}}\,dx = \frac{5}{3}\sqrt{3x^2 - 9} - \frac{1}{\sqrt{3}}\log(x\sqrt{3} + \sqrt{3x^2 - 9}) + C.$

42. $\int \frac{dy}{\sqrt{1 + y + y^2}} = \log\left(y + \frac{1}{2} + \sqrt{y^2 + y + 1}\right) + C.$

43. $\int \frac{dx}{x^2 + 3x + 1} = \frac{1}{\sqrt{5}} \log \frac{2x + 3 - \sqrt{5}}{2x + 3 + \sqrt{5}} + C.$

44. $\int \frac{dt}{2t^2 - 2t + 1} = \text{arc tan }(2t - 1) + C.$

45. $\int \frac{x^2\,dx}{\sqrt{x^3 - 9x^6}} = \frac{1}{9} \text{arc vers } 18x^3 + C.$

46. Find each of the following integrals and verify each result by differentiation:

(a) $\int \frac{dx}{9 - 16x^2}$;

(b) $\int \frac{dx}{\sqrt{9 - 16x^2}}$;

(c) $\int \frac{dx}{9 + 16x^2}$;

(d) $\int \frac{dx}{\sqrt{9 + 16x^2}}$;

(e) $\int \frac{dx}{2 - 3x^2}$;

(f) $\int \frac{dx}{\sqrt{2 - 3x^2}}$;

(g) $\int \frac{dx}{x\sqrt{4x^2 - 16}}$;

(h) $\int \frac{3\,dx}{\sqrt{5x^2 + 1}}$;

(i) $\int \frac{3x - 4}{x^2 + 9}\,dx$;

(j) $\int \frac{2x + 3}{\sqrt{9x^2 + 1}}\,dx$;

(k) $\int \frac{3x + 2}{2x^2 - 3}\,dx$;

(l) $\int \frac{dx}{x^2 + 6x + 5}$;

(m) $\int \frac{dx}{\sqrt{x^2 + 6x - 5}}$;

(n) $\int \frac{2x + 1}{\sqrt{x^2 + 6x + 7}}\,dx$;

(o) $\int \frac{3x - 2}{7x - 4x^2 + 5}\,dx$;

(p) $\int \frac{ax + b}{\sqrt{a^2x^2 - b^2}}\,dx$;

(q) $\int \frac{\sin x\,dx}{\sqrt{4 - 9\cos^2 x}}$;

(r) $\int \frac{\cos x\,dx}{9 + 4\sin^2 x}$;

(s) $\int \frac{\sec^2 x\,dx}{9 - 4\tan^2 x}$;

(t) $\int \frac{e^{ax}}{e^{2ax} + 1}\,dx$;

(u) $\int \frac{e^{ax}\,dx}{\sqrt{e^{2ax} - 1}}$;

(v) $\int \frac{dx}{x\sqrt{9 - (\log x)^2}}$;

(w) $\int \frac{x^3 \cos x^4}{1 + \sin^2 x^4}\,dx$;

(x) $\int \frac{\sin x \cos x\,dx}{9 + 4\sin^4 x}$.

II. TRIGONOMETRIC INTEGRALS

112. Introduction. Many integrals involving the trigonometric functions can be evaluated although they do not fall immediately under any one of the standard forms appearing in Art. 108. We shall now note various types of such integrals, showing by illustration how each may be integrated.

113. The type $\int \sin^m x \cos^n x\, dx$ in which m or n (or both) is a positive odd integer.

Example 1. Find $\int \sin^2 x \cos^5 x\, dx$.

SOLUTION. $\int \sin^2 x \cos^5 x\, dx = \int \sin^2 x \cos^4 x \cos x\, dx$

$$= \int \sin^2 x(1 - \sin^2 x)^2 \cos x\, dx$$

$$= \int (\sin^2 x - 2 \sin^4 x + \sin^6 x) \cos x\, dx$$

$$= \int (\sin x)^2 \cos x\, dx - 2\int (\sin x)^4 \cos x\, dx + \int (\sin x)^6 \cos x\, dx$$

$$= \tfrac{1}{3} \sin^3 x - \tfrac{2}{5} \sin^5 x + \tfrac{1}{7} \sin^7 x + C. \qquad \text{By [4], Art. 108}$$

Example 2. Find $\int \sin^3 x\, dx$.

SOLUTION. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1 - \cos^2 x) \sin x\, dx$

$$= \int \sin x\, dx + \int \cos^2 x(-\sin x\, dx)$$

$$= -\cos x + \tfrac{1}{3} \cos^3 x + C. \qquad \text{By [4], Art. 108}$$

EXERCISES

Perform and verify each of the following integrations:

1. $\int \sin^2 x \cos^3 x\, dx = \tfrac{1}{3} \sin^3 x - \tfrac{1}{5} \sin^5 x + C.$

2. $\int \cos^3 x\, dx = \sin x - \dfrac{\sin^3 x}{3} + C.$

3. $\int \cos^4 x \sin^3 x\, dx = -\tfrac{1}{5} \cos^5 x + \tfrac{1}{7} \cos^7 x + C.$

4. $\int \sin^4 x \cos^3 x\, dx = \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + C.$

5. $\int \dfrac{\cos^3 x}{\sin^4 x}\, dx = \csc x - \dfrac{1}{3} \csc^3 x + C.$

6. $\int \dfrac{\sin^3 x}{\cos^2 x}\, dx = \sec x + \cos x + C.$

7. $\int \sin^5 x\, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C.$

8. $\int \cos^5 x\, dx = \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C.$

9. $\int \sin^{\frac{2}{3}} x \cos^3 x\, dx = \frac{3}{5}\sin^{\frac{5}{3}} x - \frac{3}{11}\sin^{\frac{11}{3}} x + C.$

10. $\int \frac{\sin^5 x}{\sqrt{\cos x}}\, dx = -2\sqrt{\cos x}\left(1 - \frac{2}{5}\cos^2 x + \frac{1}{9}\cos^4 x\right) + C.$

11. $\int \frac{\sin^3 5\theta}{\cos^2 5\theta}\, d\theta = \frac{1}{5}\left[1 - \frac{1}{\cos^2 5\theta}\right]\cos 5\theta + C = -\frac{1}{5}\frac{\sin^2 5\theta}{\cos 5\theta} + C.$

12. $\int \frac{\cos^3 4y}{\sin 4y}\, dy = \frac{1}{4}\log\sin 4y - \frac{1}{8}\sin^2 4y + C.$

13. $\int \cos^7 x\, dx = \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$

14. $\int \sin^3 x \cos^3 x\, dx = -\frac{1}{6}\sin^2 x \cos^4 x - \frac{1}{12}\cos^4 x + C.$

15. Find each of the following integrals and verify each result by differentiation:

(a) $\int \cos^3 3\theta\, d\theta$;

(b) $\int \sin^3 \frac{\theta}{m}\, d\theta$;

(c) $\int \sin^3 2x \cos 2x\, dx$;

(d) $\int \sin^3 mx \cos^2 mx\, dx$;

(e) $\int \cos^5 ax \sin^2 ax\, dx$;

(f) $\int \sin^7 x \cos^2 x\, dx$;

(g) $\int \frac{\sin^3 x}{\cos^5 x}\, dx$;

(h) $\int \frac{\cos^5 mx}{\sin^3 mx}\, dx$;

(i) $\int \sin^2 x^2 \cos^3 x^2 \quad x\, dx$;

(j) $\int \sqrt{\cos x}\sin^3 x\, dx$;

(k) $\int \cos^3 x \sqrt[m]{\sin^n x}\, dx$;

(l) $\int \frac{\sin^2 (\log x) \cos^3 (\log x)\, dx}{x}$.

114. The type $\int \sin^m x \cos^n x\, dx$ in which m or n are both positive even integers.

Integrals of this type may always be evaluated by means of the following three standard trigonometric formulas (see page 359):

$$\sin x \cos x = \tfrac{1}{2}\sin 2x,$$

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x, \quad \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x.$$

Example 1. Find $\int \sin^2 x\, dx$.

SOLUTION.
$$\int \sin^2 x\, dx = \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\, dx$$
$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C.$$

Example 2. Find $\int \sin^4 x \cos^2 x \, dx.$

SOLUTION.
$$\begin{aligned}\int \sin^4 x \cos^2 x \, dx &= \int (\sin x \cos x)^2 \sin^2 x \, dx \\ &= \int \tfrac{1}{4} \sin^2 2\,x(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\,x) \, dx \\ &= \tfrac{1}{8}\int \sin^2 2\,x \, dx - \tfrac{1}{8}\int \sin^2 2\,x \cos 2\,x \, dx \\ &= \tfrac{1}{8}\int (\tfrac{1}{2} - \tfrac{1}{2}\cos 4\,x) \, dx - \tfrac{1}{8}\int \sin^2 2\,x \cos 2\,x \, dx \\ &= \tfrac{1}{16} x - \tfrac{1}{64}\sin 4\,x - \tfrac{1}{48}\sin^3 2\,x + C.\end{aligned}$$

115. The types $\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$ in which m and n have any values whatsoever except that m shall not be equal to n.

Integrals of these types are readily evaluated by means of the following three standard trigonometric formulas

$$\begin{aligned}\sin x \cos y &= \tfrac{1}{2}\sin(x + y) + \tfrac{1}{2}\sin(x - y), \\ \sin x \sin y &= -\tfrac{1}{2}\cos(x + y) + \tfrac{1}{2}\cos(x - y), \\ \cos x \cos y &= \tfrac{1}{2}\cos(x + y) + \tfrac{1}{2}\cos(x - y).\end{aligned}$$

Example. Find $\int \sin 5\,x \cos 3\,x \, dx.$

SOLUTION.
$$\begin{aligned}\int \sin 5\,x \cos 3\,x \, dx &= \int [\tfrac{1}{2}\sin(5\,x + 3\,x) + \tfrac{1}{2}\sin(5\,x - 3\,x)] \, dx \\ &= \tfrac{1}{2}\int \sin 8\,x \, dx + \tfrac{1}{2}\int \sin 2\,x \, dx \\ &= -\tfrac{1}{16}\cos 8\,x - \tfrac{1}{4}\cos 2\,x + C.\end{aligned}$$

EXERCISES

Perform and verify each of the following integrations:

1. $\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin 2\,x + C.$

2. $\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{1}{32}\sin 4\,x + C.$

3. $\int \sin^4 x \, dx = \frac{3}{8} x - \frac{1}{4}\sin 2\,x + \frac{1}{32}\sin 4\,x + C.$

4. $\int \cos^4 x \, dx = \frac{3}{8} x + \frac{1}{4}\sin 2\,x + \frac{1}{32}\sin 4\,x + C.$

5. $\int \sin 5\,x \sin 6\,x \, dx = -\frac{1}{22}\sin 11\,x + \frac{1}{2}\sin x + C.$

6. $\int \sin 3\,x \cos 5\,x \, dx = -\frac{1}{16}\cos 8\,x + \frac{1}{4}\cos 2\,x + C.$

7. $\int \cos 5x \cos 2x \, dx = \frac{1}{14} \sin 7x + \frac{1}{6} \sin 3x + C.$

8. $\int \sin^6 x \, dx = \frac{1}{16}(5x - 4 \sin 2x + \frac{1}{3} \sin^3 2x + \frac{3}{4} \sin 4x) + C.$

9. $\int \cos^6 x \, dx = \frac{1}{16}(5x + 4 \sin 2x - \frac{1}{3} \sin^3 2x + \frac{3}{4} \sin 4x) + C.$

10. $\int \sin^4 x \cos^4 x \, dx = \frac{1}{128}(3x - \sin 4x + \frac{1}{8} \sin 8x) + C.$

11. $\int \sin^8 x \, dx = \frac{1}{16}(\frac{35}{8} x - 4 \sin 2x + \frac{2}{3} \sin^3 2x + \frac{7}{8} \sin 4x + \frac{1}{64} \sin 8x) + C.$

12. Find each of the following integrals and verify each result by differentiation:

(*a*) $\int \sin^2 x \cos^4 x \, dx$; (*d*) $\int \sin^2 x \cos^6 x \, dx$;

(*b*) $\int \sin 5x \cos 7x \, dx$; (*e*) $\int \sin^6 x \cos^2 x \, dx$;

(*c*) $\int \sin 4x \cos 3x \, dx$; (*f*) $\int \cos^{10} x \, dx.$

116. The types $\int \tan^n x \, dx$ and $\int \cot^n x \, dx$, n being an integer. Integrals of these types may usually be integrated by means of the familiar trigonometric relations

$$\tan^2 x = \sec^2 x - 1, \quad \cot^2 x = \csc^2 x - 1.$$

Example. Evaluate $\int \tan^4 x \, dx$.

SOLUTION.
$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x(\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int (\tan x)^2 \, d(\tan x) - \int (\sec^2 x - 1) \, dx \\ &= \tfrac{1}{3} \tan^3 x - \tan x + x + C.\end{aligned}$$

117. The types $\int \sec^n x \, dx$ and $\int \csc^n x \, dx$, n being any even integer. Integrals of these types may also be integrated by means of the trigonometric relations noted in Art. 116.

Example. Find $\int \sec^4 x \, dx$.

SOLUTION.
$$\begin{aligned}\int \sec^4 x \, dx &= \int (\tan^2 x + 1) \sec^2 x \, dx \\ &= \int (\tan x)^2 \sec^2 x \, dx + \int \sec^2 x \, dx \\ &= \tfrac{1}{3} \tan^3 x + \tan x + C.\end{aligned}$$

118. The types $\int \tan^m x \sec^n x \, dx$ and $\int \cot^m x \csc^n x \, dx$, n being a positive even integer or m an odd integer (or both).

The cases in which n is a positive even integer may be treated by methods similar to those employed in Arts. 116 and 117.

Example. Find $\int \tan^4 x \sec^4 x \, dx$.

SOLUTION.
$$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$$
$$= \int \tan^6 x \sec^2 x \, dx + \int \tan^4 x \sec^2 x \, dx$$
$$= \tfrac{1}{7} \tan^7 x + \tfrac{1}{5} \tan^5 x + C.$$

The cases in which m is an odd integer may be treated by the method illustrated in the following example:

Example. Find $\int \tan^5 x \sec^3 x \, dx$.

SOLUTION.
$$\int \tan^5 x \sec^3 x \, dx = \int (\tan^4 x \sec^2 x)(\sec x \tan x \, dx)$$
$$= \int (\sec^2 x - 1)^2 \sec^2 x (\sec x \tan x \, dx)$$
$$= \int (\sec^6 x - 2 \sec^4 x + \sec^2 x) \, d(\sec x)$$
$$= \tfrac{1}{7} \sec^7 x - \tfrac{2}{5} \sec^5 x + \tfrac{1}{3} \sec^3 x + C. \quad \text{By [4]}$$

EXERCISES

Perform and verify each of the following integrations:

1. $\int \tan^2 x \, dx = \tan x - x + C.$ **2.** $\int \cot^2 x \, dx = -\cot x - x + C.$

3. $\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x + \log \cos x + C.$

4. $\int \cot^3 x \, dx = -\frac{1}{2} \cot^2 x - \log \sin x + C.$

5. $\int \csc^4 x \, dx = -\frac{1}{3} \cot^3 x - \cot x + C.$

6. $\int \sec^6 x \, dx = \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C.$

7. $\int \csc^6 x \, dx = -\frac{1}{5} \cot^5 x - \frac{2}{3} \cot^3 x - \cot x + C.$

8. $\int \tan^4 x \sec^4 x \, dx = \frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x + C.$

9. $\int \tan^3 x \sec^5 x \, dx = -\frac{1}{5} \sec^5 x + \frac{1}{7} \sec^7 x + C.$

10. $\int \cot^5 x \csc^3 x \, dx = -\frac{1}{7} \csc^7 x + \frac{2}{5} \csc^5 x - \frac{1}{3} \csc^3 x + C.$

11. $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log \cos x + C.$

12. $\int \cot^5 x\, dx = -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log\sin x + c.$

13. $\int \tan x \sec^4 x\, dx = \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + C.$

14. $\int \tan^3 x \sec^{\frac{2}{3}} x\, dx = \frac{3}{8}\sec^{\frac{8}{3}} x - \frac{3}{2}\sec^{\frac{2}{3}} x + C.$

15. $\int \frac{\sec^4 x}{\tan^2 x}\, dx = \tan x - \frac{1}{\tan x} + C.$

16. $\int (\tan x + \cot x)^2\, dx = \tan x - \cot x + C.$

17. Find each of the following integrals and verify each result by differentiation:

(*a*) $\int \tan^2 x \sec^2 x\, dx$; (*e*) $\int \frac{dx}{\cot^3 x}$; (*i*) $\int \tan^2 x \sec^4 x\, dx$;

(*b*) $\int \tan^3 x \sqrt{\sec x}\, dx$; (*f*) $\int \frac{\tan^5 x}{\cos x}\, dx$; (*j*) $\int \frac{\sec^4 x}{\tan^3 x}\, dx$;

(*c*) $\int \frac{\csc^4 ax}{\cot^3 ax}\, dx$; (*g*) $\int \frac{\sqrt{\tan x}}{\cos^2 x}\, dx$; (*k*) $\int \frac{\sec^4 x}{\tan^5 x}\, dx$;

(*d*) $\int \frac{dx}{\tan^4 x}$; (*h*) $\int \frac{\cot^{\frac{3}{4}} x}{\sin^2 x}\, dx$; (*l*) $\int \sec^8 x\, dx$.

III. THE INTEGRATION OF RATIONAL FRACTIONS

119. Introduction. An integral rational fraction in x is a fraction whose numerator and denominator contain x to positive integral powers only; that is, both numerator and denominator are polynomials in x. In case the degree of the numerator is equal to or higher than that of the denominator, such a fraction may be converted by ordinary division into a polynomial quotient plus a remainder which is in the form of a rational fraction having the degree of its numerator less than that of its denominator. The study of the integration of such fractions may therefore be confined to fractions having the degree of their numerator less than that of their denominator.

For example, by division such as in elementary algebra we may write

$$\frac{x^4 + 3x^3 + x^2 + 1}{x^2 + 2x + 3} = x^2 + x - 4 + \frac{5x + 13}{x^2 + 2x + 3}$$

and therefore we have

$$\int \frac{x^4 + 3x^3 + x^2 + 1}{x^2 + 2x + 3}\, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 4x + \int \frac{5x + 13}{x^2 + 2x + 3}\, dx.$$

It thus remains but to evaluate the last term; that is, to integrate a rational fraction whose numerator, $5x + 13$, is of lesser degree than its denominator, $x^2 + 2x + 3$.

As thus limited, the general study of integrating rational fractions may be conveniently considered under three cases as follows:

120. Case I. *When the denominator may be factored into real linear factors, none of which are repeated.*

The following example is typical of this case and of the method of integrating a rational fraction under such circumstances:

Example. Find $\int \frac{3x + 2}{x(x+1)(x-2)}\,dx.$

SOLUTION. Inasmuch as the factors of the denominator are x, $x + 1$, and $x - 2$ it is possible (as shown in algebra) to express the fraction in the form:

$$\frac{3x+2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}, \tag{1}$$

where A, B, and C are constants as yet undetermined. In order to find the actual values of A, B, and C, clear (1) of fractions, thus obtaining

$$3x + 2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1), \tag{2}$$

or, performing the indicated multiplications on the right and arranging the result in descending powers of x,

$$3x + 2 = (A + B + C)x^2 + (-A - 2B + C)x - 2A. \tag{3}$$

This relation, like (1) and (2), holds for *all* values of x. Hence (as shown in algebra) the coefficients of like powers of x on the two sides of the relation must be equal; that is, we have the following set of equations for determining A, B, and C:

$$\left.\begin{aligned} A + B + C &= 0, \\ -A - 2B + C &= 3, \\ -2A \qquad &= 2. \end{aligned}\right\} \tag{4}$$

Upon solving (4) in the usual manner of elementary algebra, we find that $A = -1$, $B = -\frac{1}{3}$, $C = \frac{4}{3}$. Placing these values in (1), we have

$$\frac{3x+2}{x(x+1)(x-2)} = -\frac{1}{x} - \frac{1}{3(x+1)} + \frac{4}{3(x-2)}.$$

Hence,

$$\int \frac{3x+2}{x(x+1)(x-2)}\,dx = -\int \frac{dx}{x} - \frac{1}{3}\int \frac{dx}{x+1} + \frac{4}{3}\int \frac{dx}{x-2}$$
$$= -\log x - \tfrac{1}{3}\log(x+1) + \tfrac{4}{3}\log(x-2) + C.$$

Finally, this result may be written in condensed form by writing the arbitrary constant C in the form $\log C$ and subsequently employing well-known principles of logarithms. Thus, we may write

$$\int \frac{3x+2}{x(x+1)(x-2)}\,dx = -\log x - \log(x+1)^{\frac{1}{3}} + \log(x-2)^{\frac{4}{3}} + \log C$$
$$= \log \frac{C(x-2)^{\frac{4}{3}}}{x(x+1)^{\frac{1}{3}}}.$$

Note. A short method of determining the constants A, B, C without solving the system (4) is as follows: Inasmuch as (2) holds for all values of x, suppose in particular that $x = 0$. Then (2) reduces to $2 = A \quad (-2)$, giving at once $A = -1$. Next suppose $x = -1$. Then (2) gives $-3 + 2 = B \quad (-1)(-3)$, or $-1 = 3B$. Hence $B = -\frac{1}{3}$. Finally, place $x = 2$. The result in (2) is $6 + 2 = C \quad 2 \quad 3$ or $8 = 6C$. Hence $C = \frac{4}{3}$.

EXERCISES

Establish each of the following results:

1. $\int \frac{dx}{4 - x^2} = \int \frac{dx}{(2 - x)(2 + x)} = \frac{1}{4} \log \frac{2 + x}{2 - x} + C.$

2. $\int \frac{x\,dx}{4 - x^2} = -\frac{1}{2} \log (4 - x^2) + C.$

3. $\int \frac{x^2\,dx}{4 - x^2} = -x + \log \frac{x + 2}{x - 2} + C.$ HINT. See Art. 119.

4. $\int \frac{7x - 1}{(1 - 3x)(1 - 2x)}\,dx = -\frac{4}{3} \log (1 - 3x) + \frac{5}{2} \log (1 - 2x) + C.$

5. $\int \frac{5x + 6}{2 - x - x^2}\,dx = -\frac{4}{3} \log (2 + x) - \frac{11}{3} \log (1 - x) + C.$

6. $\int \frac{2x^3 + 1}{x^2 + 3x + 2}\,dx = x^2 - 6x + 15 \log (x + 2) - \log (x + 1) + C.$

7. $\int \frac{1 + 3x}{1 + 11x + 28x^2}\,dx = \frac{4}{21} \log (1 + 7x) - \frac{1}{12} \log (1 + 4x) + C.$

8. $\int \frac{2x - 4}{(1 - x^2)(1 - 2x)}\,dx = 2 \log (1 - 2x) - \log (1 - x^2) + C.$

9. $\int \frac{x^2 + 7x + 3}{x^2 + 7x + 10}\,dx = x + \frac{7}{3} \log \frac{x + 5}{x + 2} + C.$

10. $\int \frac{1 + 3x + 2x^2}{(1 - 2x)(1 - x^2)}\,dx = 3 \log (1 - x) - 2 \log (1 - 2x) + C.$

11. $\int \frac{x^2 - 10x + 13}{(x - 1)(x^2 - 5x + 6)}\,dx = \log \frac{(x - 1)^2(x - 2)^3}{(x - 3)^4} + C.$

12. $\int \frac{2x^3 + x^2 - x - 3}{x(x - 1)(2x + 3)}\,dx = x + \log x - \frac{1}{5} \log (x - 1) - \frac{4}{5} \log (2x + 3) + C.$

13. $\int \frac{mx + n}{(x - a)(x + b)}\,dx$

$$= \frac{1}{a + b} [(ma + n) \log (x - a) + (mb - n) \log (x + b)] + C.$$

14. Find each of the following integrals:

(a) $\int \frac{x + 2}{x(x + 1)}\,dx;$

(b) $\int \frac{3x - 1}{x^2 - x - 6}\,dx;$

(c) $\int \frac{5x - 11}{2x^2 + x - 6}\,dx;$

(d) $\int \frac{x^4 + x^3 - 3}{x^3 - 2x}\,dx;$

(e) $\int \frac{2x^2 - 1}{(x^2 - 1)(x - 3)(x + 2)}\,dx;$

(f) $\int \frac{dx}{(x^2 - 4)(x^2 - 9)}.$

121. Case II. *When the factors of the denominator are all of the first degree and some repeated.*

The following example is typical of this case and of the method of integrating the fraction under such circumstances:

Example. Find $\int \frac{2x+1}{x(x+1)^3}dx.$

SOLUTION. Inasmuch as the factors of the denominator are x and $(x + 1)$, of which x occurs but once while $(x + 1)$ is repeated three times, it is possible (as shown in advanced algebra) to express the fraction in the following form:

$$\frac{2x+1}{x(x+1)^3} = \frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1}, \tag{1}$$

where A, B, C, D are constants as yet undetermined. In order to find the actual values of these constants, clear (1) of fractions, obtaining

$$2x + 1 = A(x+1)^3 + Bx + Cx(x+1) + Dx(x+1)^2, \tag{2}$$

or, expanding and arranging the second member in descending powers of x,

$$2x + 1 = (A + D)x^3 + (3A + C + 2D)x^2 + (3A + B + C + D)x + A. \tag{3}$$

This relation, like (1) and (2), holds for *all* values of x. Hence, as shown in algebra, the coefficients of like powers of x must be equal; that is, we have the following set of equations for determining A, B, C, and D.

$$A + D = 0,\ \ 3A + C + 2D = 0,\ \ 3A + B + C + D = 2,\ \ A = 1. \tag{4}$$

Upon solving (4) in the usual manner, we find $A = 1$, $B = 1$, $C = -1$, $D = -1$. Placing these values in (1), we obtain

$$\frac{2x+1}{x(x+1)^3} = \frac{1}{x} + \frac{1}{(x+1)^3} - \frac{1}{(x+1)^2} - \frac{1}{x+1}.$$

Hence

$$\int \frac{2x+1}{x(x+1)^3}dx = \log x - \frac{1}{2(x+1)^2} + \frac{1}{x+1} - \log(x+1) + C$$
$$= \frac{1+2x}{2(x+1)^2} + \log\frac{x}{x+1} + C.$$

Note 1. In order to determine the constants A, B, C, D in the foregoing example without solving the system (4) one may proceed as follows (compare Note, Art. 120): Place $x = 0$ in (2), obtaining at once $A = 1$. Then place $x = -1$ in (2), obtaining $2(-1) + 1 = B(-1)$ from which $B = 1$. Now differentiate both members of (2) and in the result place in succession $x = 0$ and $x = -1$. If the values already found for A and B be introduced, we then obtain the two equations $C + D = -2$, $C = -1$. Hence, $C = D = -1$.

Note 2. In employing the method of expressing a fraction as the sum of simple so-called "partial fractions," as illustrated in (1), it is to be noted that to every n-fold linear factor in the denominator, such as $(x - a)^n$, there corresponds a sum of n partial fractions of the type

$$\frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \frac{C}{(x-a)^{n-2}} + \quad + \frac{M}{(x-a)^2} + \frac{N}{(x-a)}.$$

Thus, in the foregoing example if the fraction to be integrated had been $\dfrac{2x+1}{x^2(x+1)^3}$, the relation corresponding to (1) would have been

$$\frac{2x+1}{x^2(x+1)^3} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)}.$$

In all cases the constants A, B, C, etc. may be determined either through a system of equations analogous to (3) or by some special device such as suggested in Note 1.

EXERCISES

Establish each of the following results:

1. $\displaystyle\int \frac{9\,dx}{(x-1)(x+2)^2} = \frac{3}{(x+2)} - \log\frac{x+2}{x-1} + C.$

2. $\displaystyle\int \frac{3x^2-2}{(x+2)^3}\,dx = \frac{12x+19}{(x+2)^2} + 3\log(x+2) + C.$

3. $\displaystyle\int \frac{4+7x}{(2+3x)(1+x)^2}\,dx = 2\log\frac{1+x}{2+3x} - \frac{3}{1+x} + C.$

4. $\displaystyle\int \frac{5x^2+x-10}{x^2(2x+5)}\,dx = \log x + \frac{3}{2}\log(2x+5) + \frac{2}{x} + C.$

5. $\displaystyle\int \frac{30x^2+43x-8}{(x+4)(3x+2)^2}\,dx = \log(x+4)^3\sqrt[3]{3x+2} + \frac{7}{3(3x+2)} + C.$

6. $\displaystyle\int \frac{x^2-10x-5}{x(2x^2+3x-5)}\,dx = \log\frac{x(2x+5)^{\frac{3}{2}}}{(x-1)^2} + C.$

7. $\displaystyle\int \frac{x^2+pq}{x(x-p)(x+q)}\,dx = \log\frac{(x-p)(x+q)}{x} + C.$

8. $\displaystyle\int \frac{1-x+2x^2}{(1-x)^3}\,dx = \frac{1}{(1-x)^2} - \frac{3}{1-x} - 2\log(1-x) + C.$

9. $\displaystyle\int \frac{(x-1)^2}{(x+1)^2(x+3)^2}\,dx = -\frac{5x+7}{x^2+4x+3} + 2\log\left(\frac{x+3}{x+1}\right) + C.$

10. $\displaystyle\int \frac{3x-4}{(x-2)(x-1)^3}\,dx = \frac{4x-5}{2(x-1)^2} + 2\log\frac{x-2}{x-1} + C.$

11. Find each of the following integrals:

(a) $\displaystyle\int \frac{x}{(x-1)^2(x+2)}\,dx;$ (d) $\displaystyle\int \frac{9x^4}{x^3-2x^2+x}\,dx;$

(b) $\displaystyle\int \frac{3x^2-4}{x^3+x^2}\,dx;$ (e) $\displaystyle\int \frac{x^2+a}{x(x-a)^2}\,dx;$

(c) $\displaystyle\int \frac{dx}{x^4-x^2};$ (f) $\displaystyle\int \frac{x^2+ab}{x(x-a)^2(x-b)^2}\,dx.$

122. Case III. *When the denominator contains one or more factors of the second degree, but none of them repeated.*

The following example is typical of this case and of the method of integrating the fraction under such circumstances:

Example. Find $\int \frac{7 + x}{(1 + x)(1 + x^2)} dx.$

SOLUTION. Inasmuch as the factors of the denominator are $(1 + x)$ and $(1 + x^2)$ it is possible (as shown in advanced algebra) to express the fraction in the following form:

$$\frac{7 + x}{(1 + x)(1 + x^2)} = \frac{A}{1 + x} + \frac{Bx + C}{1 + x^2}, \tag{1}$$

where A, B, and C are constants yet to be determined. In order to find the actual values of A, B, C, clear (1) of fractions, obtaining

$$7 + x = A(1 + x^2) + (Bx + C)(1 + x),$$

or

$$7 + x = (A + B)x^2 + (B + C)x + (A + C).$$

Thus, A, B, and C are such that

$$A + B = 0, \qquad B + C = 1, \qquad A + C = 7.$$

Solving, we find $A = 3$, $B = -3$, $C = 4$. Relation (1) thus becomes

$$\frac{7 + x}{(1 + x)(1 + x^2)} = \frac{3}{1 + x} + \frac{-3x + 4}{1 + x^2} = \frac{3}{1 + x} - 3\frac{x}{1 + x^2} + \frac{4}{1 + x^2}.$$

Integrating, we obtain as the desired result

$$\int \frac{7 + x}{(1 + x)(1 + x^2)} dx = 3 \log (1 + x) - \frac{3}{2} \log (1 + x^2) + 4 \arctan x + C.$$

Note 1. In employing the method illustrated in the foregoing example it is to be observed that, in general, to every non-repeated second-degree factor in the denominator, that is, to every factor of the typical form $ax^2 + bx + c$, there corresponds a partial fraction of the form

$$\frac{Mx + N}{ax^2 + bx + c}; \quad M, N = \textit{constants}.$$

Thus, if the fraction to be integrated in the foregoing example had been $\frac{7 + x}{(1 + x)(1 + x^2)(2x^2 + x + 3)}$, the relation corresponding to (1) would be

$$\frac{7 + x}{(1 + x)(1 + x^2)(2x^2 + x + 3)} = \frac{A}{1 + x} + \frac{Bx + C}{1 + x^2} + \frac{Dx + E}{2x^2 + x + 3},$$

where A, B, C, D, E are constants to be determined.

Note 2. The case in which the given fraction has a denominator containing one or more second-degree factors, some of which are repeated (compare Art. 121), may be treated by methods analogous to the foregoing, but this case is here omitted because of its complications and its relative unimportance.

EXERCISES

Establish each of the following results:

1. $\int \frac{8\,dx}{x(x^2+4)} = \log x^2 - \log(x^2+4) + C = \log \frac{Cx^2}{x^2+4}$.

2. $\int \frac{42 - 19\,x}{(x^2+1)(x-4)}\,dx = \log(x^2+1) - 2\log(x-4) - 11 \arctan x + C$.

3. $\int \frac{x^2 - 3\,x + 3}{x(x^2+3)}\,dx = \log x - \sqrt{3} \arctan \frac{x}{\sqrt{3}} + C$.

4. $\int \frac{12 - x - x^2}{(3\,x-2)(x^2+5)}\,dx = \frac{2}{3}\log(3\,x-2) - \frac{1}{2}\log(x^2+5) - \frac{1}{\sqrt{5}} \arctan \frac{x}{\sqrt{5}} + C$.

5. $\int \frac{(x+1)^2}{x^3+x}\,dx = \log x + 2 \arctan x + C$.

6. $\int \frac{x^3 - 1}{x^3 + 3\,x}\,dx = x + \frac{1}{6}\log\frac{x^2+3}{x^2} - \sqrt{3} \arctan \frac{x}{\sqrt{3}} + C$.

7. $\int \frac{2\,x^2 + 3\,x + 6}{x^3 + 3\,x}\,dx = \log x^2 + \sqrt{3} \arctan \frac{x}{\sqrt{3}} + C$.

8. $\int \frac{7\,x^2 + 9}{x^3 + 3\,x}\,dx = \log Cx^3(x^2+3)^2$.

9. $\int \frac{dx}{x^3+1} = \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \arctan \frac{2\,x-1}{\sqrt{3}} + C$.

10. $\int \frac{dx}{(x^2+1)(x^2+x)} = \frac{1}{4}\log\frac{x^4}{(x+1)^2(x^2+1)} - \frac{1}{2} \arctan x + C$.

11. Find each of the following integrals:

(a) $\int \frac{dx}{x(x^2+1)}$; (c) $\int \frac{3\,x+2}{x(x^2+5\,x+1)}\,dx$;

(b) $\int \frac{3+x}{(x-1)(x^2+4)}\,dx$; (d) $\int \frac{dx}{ax^2+bx+c}$;

IV Integration by Parts

123. Method. Suppose that u and v are any two given functions of x. Then, by Ex. 8, page 202, we have

$$d(u \cdot v) = u\,dv + v\,du,$$

which may be written in the form

$$u\,dv = d(u \cdot v) - v\,du.$$

When stated in the language of integration instead of differentiation, the last equation is evidently equivalent to the following:

$$\int u\,dv = uv - \int v\,du.$$

This is known as the formula for **integration by parts.** The manner in which it may be employed actually to determine the integral of a given function of x, at least in various cases, will now be illustrated.

Example 1. Find $\int x \cos x \, dx$.

SOLUTION. Let $u = x$ and $dv = \cos x \, dx$.

Then $du = dx$ and $v = \int \cos x \, dx = \sin x$.

Substituting in the foregoing formula, we obtain

$$\int \overbrace{x}^{u} \overbrace{\cos x \, dx}^{dv} = \overbrace{x}^{u} \overbrace{\sin x}^{v} - \int \overbrace{\sin x}^{v} \overbrace{dx}^{du}$$
$$= x \sin x + \cos x + C.$$

Example 2. Find $\int \log x \, dx$.

SOLUTION. Let $u = \log x$ and $dv = dx$.

Then $du = \dfrac{dx}{x}$ and $v = \int dx = x$.

Hence, substituting in the formula, we obtain

$$\int \log x \, dx = x \log x - \int x \cdot \frac{dx}{x}$$
$$= x \log x - \int dx = x \log x - x + C.$$

Example 3. Find $\int x^2 e^x \, dx$.

SOLUTION. Let $u = x^2$ and $dv = e^x \, dx$.

Then $du = 2\,x \, dx$ and $v = \int e^x \, dx = e^x$.

Substituting in the formula, we obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx. \qquad (1)$$

The integral here appearing in the second member may now be found by applying the same process again. Thus, if we now take $u = x$ and $dv = e^x \, dx$, so that $du = dx$, $v = e^x$, we obtain

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C. \qquad (2)$$

From (1) and (2) we thus reach the final result, namely,

$$\int x^2 e^x \, dx = x^2 e^x - 2\,x e^x + 2\,e^x + C.$$

This example illustrates how the process of integration by parts may, if desired, be used more than once in evaluating a given integral.

An examination of the foregoing illustrations shows that in employing the formula for integration by parts to find any given integral, as $\int f(x) \, dx$, it is necessary to identify $f(x) \, dx$ as some product of

the form, $u\,dv$, where u and dv are of such a nature that (1) $\int dv$ can be readily determined, thus giving v itself, and (2) $\int v\,du$ is readily integrable. No general rule can be given for determining such a u and dv for a given $f(x)\,dx$, nor is it to be inferred that they even exist in all cases. On the other hand, such a u and dv can be found in more than one way in some cases. Despite this uncertainty, it is a familiar fact that experience in solving a variety of simple exercises involving the principle soon makes one able to supply the necessary ingenuity in more complicated cases and leads in the end to a wide usefulness of the principle.

EXERCISES

Establish each of the following results by means of one or more integrations by parts:

1. $\int xe^{3x}\,dx = \frac{1}{3}e^{3x}(x - \frac{1}{3}) + C.$

HINT. Note that one may here advantageously choose $u = x$, $dv = e^{3x}\,dx$.

2. $\int x \sin x\,dx = -x\cos x + \sin x + C.$

3. $\int x \log x\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4} + C.$

4. $\int x^n \log x\,dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C.$

5. $\int \text{arc}\sin x\,dx = x\,\text{arc}\sin x + \sqrt{1-x^2} + C.$

6. $\int \text{arc}\tan x\,dx = x\,\text{arc}\tan x - \frac{1}{2}\log(1+x^2) + C.$

7. $\int x\,\text{arc}\tan x\,dx = \frac{x^2+1}{2}\text{arc}\tan x - \frac{x}{2} + C.$

8. $\int x^2 \sin x\,dx = 2\cos x + 2x\sin x - x^2\cos x + C.$

9. $\int x^3 e^x\,dx = e^x(x^3 - 3x^2 + 6x - 6) + C.$

10. $\int x\tan^2 x\,dx = x\tan x - \frac{x^2}{2} + \log\cos x + C.$

11. $\int x^2\,\text{arc}\sin x\,dx = \frac{x^3}{3}\text{arc}\sin x + \frac{x^2+2}{9}\sqrt{1-x^2} + C.$

12. $\int (\log x)^2\,dx = x[(\log x)^2 - 2\log x + 2] + C.$

13. $\int \cos x \log\sin x\,dx = \sin x(\log\sin x - 1) + C.$

14. $\int \sec^2 x \log\tan x\,dx = \tan x(\log\tan x - 1) + C.$

15. $\int \frac{\log x}{(x+1)^2}\,dx = \frac{x}{x+1}\log x - \log(x+1) + C.$

HINT. Take $u = \log x$.

16. $\int x^3\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}x^2(a^2 - x^2)^{\frac{3}{2}} - \frac{2}{15}(a^2 - x^2)^{\frac{5}{2}} + C.$

HINT. Take $u = x^2$, $dv = \sqrt{a^2 - x^2}\,x\,dx$.

17. $\int e^{ax}\sin mx\,dx = \frac{e^{ax}(a\sin mx - m\cos mx)}{a^2 + m^2} + C.$

HINT. First take $u = e^{ax}$, $dv = \sin mx\,dx$, thus obtaining

$$\int e^{ax}\sin mx\,dx = -\frac{e^{ax}\cos mx}{m} + \frac{a}{m}\int e^{ax}\cos mx\,dx. \qquad (1)$$

Next, in the original integral take $u = \sin mx$, $dv = e^{ax}\,dx$, thus obtaining

$$\int e^{ax}\sin mx\,dx = \frac{e^{ax}\sin mx}{a} - \frac{m}{a}\int e^{ax}\cos mx\,dx. \qquad (2)$$

Now combine (1) and (2) so as to get the indicated result.

18. $\int e^{ax}\cos mx\,dx = \frac{e^{ax}(m\sin mx + a\cos mx)}{a^2 + m^2} + C.$

HINT. See Ex. 17.

19. $\int \sqrt{a^2 + x^2}\,dx = \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{a^2}{2}\log(x + \sqrt{a^2 + x^2}) + C.$

HINT. Take $u = \sqrt{a^2 + x^2}$, $dv = dx$. In the new integral encountered use the relation

$$\frac{x^2}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2} - \frac{a^2}{\sqrt{a^2 + x^2}}.$$

20. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C.$

21. Find each of the following integrals:

(a) $\int xe^{2x}\,dx$; (b) $\int \operatorname{arc\,cot} x\,dx$; (c) $\int x^2e^{-x}\,dx$;

(d) $\int \log(\log x)\frac{dx}{x}$; (e) $\int \frac{\log(x+1)}{\sqrt{x+1}}\,dx$; (f) $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}}$;

(g) $\int \frac{x^3\,dx}{\sqrt{x^2 + a^2}}$; (h) $\int \sec^5 x\,dx$; (i) $\int \tan^2 x\sec x\,dx$.

V. INTEGRATION BY SUBSTITUTION

124. Algebraic substitutions. The process of finding an integral $\int f(x)\,dx$ is often facilitated by changing the variable of integration, x, into some other variable, say z, where z bears some given algebraic relation to x. The following illustrations will convey the nature and value of this process.

Example 1. Find $\int \frac{dx}{x\sqrt{x+1}}$.

SOLUTION. Let $z = \sqrt{x+1}$.

Then $x + 1 = z^2$ and hence $x = z^2 - 1$, $dx = 2\,z\,dz$.

Substituting, $\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2\,z\,dz}{(z^2-1)z} = 2\int \frac{dz}{z^2-1} = \log \frac{z-1}{z+1} + C.$

Replacing z by its value in terms of the original variable x, we obtain as the final result

$$\int \frac{dx}{x\sqrt{x+1}} = \log \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} + C.$$

Example 2. Find $\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1}$.

SOLUTION. Let $x = z^4$, or $z = x^{\frac{1}{4}}$.

Then, $x^{\frac{1}{2}} = z^2$, $x^{\frac{3}{4}} = z^3$, and $dx = 4\,z^3\,dz$.

Substituting, $\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1} = \int \frac{4\,z^5\,dz}{z^3+1} = 4\int \left(z^2 - \frac{z^2}{z^3+1}\right) dz$

$$= 4\int z^2\,dz - 4\int \frac{z^2\,dz}{z^3+1} = \frac{4}{3}z^3 - \frac{4}{3}\log(z^3+1) + C.$$

Replacing z by its value $x^{\frac{1}{4}}$, we obtain as the final result

$$\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1} = \frac{4}{3}x^{\frac{3}{4}} - \frac{4}{3}\log(x^{\frac{3}{4}}+1) + C.$$

Example 3. Find $\int \sin\sqrt{x}\,dx$.

SOLUTION. Let $x = z^2$, or $z = \sqrt{x}$. Then $dx = 2\,z\,dz$, so that we have

$\int \sin\sqrt{x}\,dx = 2\int z\sin z\,dz = 2[-z\cos z + \sin z] + C.$ Ex. 2, page 239

Replacing z by its value $\sqrt{x}$, we obtain as the final result

$$\int \sin\sqrt{x}\,dx = 2[-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}] + C.$$

An examination of the foregoing illustrations shows that the essential step in each instance consists in making such a substitution for x as will give rise to a new integrand (or expression to be integrated) in which no radicals or fractional exponents enter. No general rule can be given to accomplish this, yet, as in the use of integration by parts (Art. 123), experience soon leads one to foresee such transformations as will be advantageous for the purpose.

125. Trigonometric substitutions. Integrals whose integrands involve fractional powers of either $a^2 - x^2$ or $a^2 + x^2$ or $x^2 - a^2$ may frequently be evaluated by introducing a suitable trigonometric substitution. In general, the following rules may be followed:

If the integrand involves $a^2 - x^2$, *substitute* $x = a \sin z$.
If the integrand involves $a^2 + x^2$, *substitute* $x = a \tan z$.
If the integrand involves $x^2 - a^2$, *substitute* $x = a \sec z$.

Example. Find $\displaystyle\int \frac{dx}{x^2(a^2 - x^2)^{\frac{1}{2}}}$.

SOLUTION. Let $x = a \sin z$.

Then, $x^2 = a^2 \sin^2 z,\ a^2 - x^2 = a^2(1 - \sin^2 z) = a^2 \cos^2 z$;

also $dx = a \cos z\, dz$.

Substituting,
$$\int \frac{dx}{x^2(a^2 - x^2)^{\frac{1}{2}}} = \int \frac{a \cos z\, dz}{a^2 \sin^2 z \quad a \cos z} = \frac{1}{a^2}\int \frac{dz}{\sin^2 z}$$
$$= \frac{1}{a^2}\int \csc^2 z\, dz = -\frac{1}{a^2}\cot z + C.$$

Now, $\sin z = \dfrac{x}{a}$. Therefore, $\cot z = \dfrac{\sqrt{a^2 - x^2}}{x}$.

For a final result we thus have
$$\int \frac{dx}{x^2(a^2 - x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$$

EXERCISES

Establish each of the following results:

1. $\displaystyle\int \frac{dx}{1 - \sqrt{x}} = -2\sqrt{x} - 2 \log(1 - \sqrt{x}) + C.$

2. $\displaystyle\int \frac{dx}{1 + \sqrt{1 + x}} = 2\sqrt{1 + x} - 2 \log(1 + \sqrt{1 + x}) + C.$

3. $\displaystyle\int \frac{x\, dx}{1 + \sqrt{x}} = \frac{2}{3}x^{\frac{3}{2}} - x + 2\sqrt{x} - 2 \log(1 + \sqrt{x}) + C.$

4. $\displaystyle\int \frac{dx}{1 + (x + 1)^{\frac{1}{3}}} = \frac{3}{2}(x + 1)^{\frac{2}{3}} - 3(x + 1)^{\frac{1}{3}} + 3 \log[1 + (x + 1)^{\frac{1}{3}}] + C.$

5. $\displaystyle\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log(x^{\frac{1}{6}} + 1) + C.$

HINT. Let $x = z^6$.

6. $\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{2}{3}} - 1} = \frac{6}{5} x^{\frac{5}{6}} + 6 x^{\frac{1}{6}} + \frac{3}{2} \log \frac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{6}} + 1} - 3 \arctan x^{\frac{1}{6}} + C.$

7. $\int \frac{dx}{x^{\frac{7}{6}} + x^{\frac{4}{3}}} = -\frac{6}{x^{\frac{1}{6}}} + \log \frac{(x^{\frac{1}{6}} + 1)^6}{x} + C.$

8. $\int \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} - 1}\,dx = x + 1 + 4\sqrt{x + 1} + 4 \log (\sqrt{x + 1} - 1) + C.$

9. $\int \frac{x\,dx}{(3x + 2)^{\frac{2}{3}}} = \frac{1}{4}(x - 2)(3x + 2)^{\frac{1}{3}} + C.$

10. $\int \frac{dx}{x + 2 + \sqrt{x + 2}} = 2 \log (\sqrt{x + 2} + 1) + C.$

11. $\int \frac{x^2\,dx}{\sqrt{1 - x}} = -\frac{2}{15}(3x^2 + 4x + 8)\sqrt{1 - x} + C.$

12. $\int \frac{x^3\,dx}{(2 + x^2)^{\frac{3}{2}}} = \frac{x^2 + 4}{\sqrt{2 + x^2}} + C.$

13. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin \frac{x}{a} + C.$

HINT. See Art. 125.

14. $\int \frac{dx}{x^2\sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C.$

15. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{x}{a} + C.$

16. $\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a \log \frac{a + \sqrt{a^2 - x^2}}{x} + C.$

17. $\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} + a \arcsin \frac{a}{x} + C.$

18. $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 + x^2}} + C.$

19. $\int \frac{x^2\,dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2 - x^2}} - \arcsin \frac{x}{a} + C.$

20. $\int \frac{dx}{x^2\sqrt{4x^2 - 1}} = \frac{\sqrt{4x^2 - 1}}{x} + C.$

21. $\int \frac{dx}{x\sqrt{a^2 - x^2}} = \frac{1}{a} \log \frac{x}{a + \sqrt{a^2 - x^2}} + C.$

22. $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C.$

23. $\int \frac{dx}{x\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax} + C.$ HINT. Let $x = 2a \sin^2 z$.

24. $\int \frac{dx}{x\sqrt{2\,ax + x^2}} = -\frac{\sqrt{2\,ax + x^2}}{ax} + C.$ HINT. Let $x = 2\,a\tan^2 z$.

25. $\int \frac{dx}{x\sqrt{x^2 - 2\,ax}} = \frac{\sqrt{x^2 - 2\,ax}}{ax} + C.$ HINT. Let $x = 2\,a\sec^2 z$.

26. Find each of the following integrals:

(*a*) $\int \frac{dx}{1 + \sqrt{x}}$;

(*b*) $\int \frac{x^2\,dx}{1 - \sqrt{x}}$;

(*c*) $\int \frac{\sqrt{x+1}}{x}\,dx$;

(*d*) $\int \frac{x^{\frac{3}{4}}}{x^{\frac{1}{2}} + x^{\frac{1}{4}}}\,dx$;

(*e*) $\int \frac{dx}{1 + x^{\frac{1}{3}}}$;

(*f*) $\int \frac{dx}{1 + (x+1)^{\frac{1}{3}}}$;

(*g*) $\int x(a + x)^{\frac{1}{3}}\,dx$;

(*h*) $\int \frac{dx}{(x+1)^{\frac{2}{3}} - (x+1)^{\frac{1}{2}}}$;

(*i*) $\int \frac{x\,dx}{(a + bx)^{\frac{3}{2}}}$;

(*j*) $\int \frac{x^2\,dx}{(x^2 - a^2)^{\frac{1}{2}}}$;

(*k*) $\int (a^2 - x^2)^{\frac{3}{2}}\,dx$;

(*l*) $\int x^2(a^2 - x^2)^{\frac{1}{2}}\,dx$;

(*m*) $\int \frac{dx}{x(a^2 + x^2)^{\frac{1}{2}}}$;

(*n*) $\int \frac{(a^2 + x^2)^{\frac{3}{2}}}{x^2}\,dx$;

(*o*) $\int \frac{x\,dx}{\sqrt{2\,ax - x^2}}$.

126. Integral tables. It is to be understood that the various integral forms which have been encountered in the present chapter represent but a small part of the total of possible integrable forms. Enough has been given, however, to convey some idea of the great variety of such forms. In order that the student may have a more complete collection systematically arranged he is advised to purchase at this point a separate "Table of Integrals." The one by B. O. Peirce, published by Ginn and Co. (Boston), is highly recommended. A brief collection may be found on pages 353–358 of the present text.

Note. It is worthy of note that the standard integral formula [16] of Art. 108 may be expressed differently as follows: From trigonometry we have

$$\sec u + \tan u = \frac{1 + \sin u}{\cos u} = \frac{1 + \sin u}{\sqrt{1 - \sin^2 u}} = \sqrt{\frac{1 + \sin u}{1 - \sin u}}$$

$$= \sqrt{\frac{1 - \cos\left(u + \frac{\pi}{2}\right)}{1 + \cos\left(u + \frac{\pi}{2}\right)}} = \tan\frac{1}{2}\left(u + \frac{\pi}{2}\right).$$

Thus, [16] takes the form

$$\int \sec u\,du = \log\tan\frac{1}{2}\left(u + \frac{\pi}{2}\right) + C.$$

Similarly, instead of [17] of Art. 108 we may write

$$\int \csc u\,du = \log\tan\frac{u}{2} + C.$$

This appears by using $u - \frac{\pi}{2}$ instead of u throughout the foregoing reasoning.

CHAPTER XIII

APPLICATIONS OF INTEGRATION

127. Introduction. In the preceding chapter we have been concerned with the finding of the integral of a given function. No applications of the integral to geometry or physics have been indicated, but such applications are immediate. This will be made clear in the present chapter.

128. Length of curve. As a first application of integration, let us determine the length of the parabola $y = \frac{1}{2}x^2$ from its vertex, which is situated at the origin (0, 0) to the point (2, 2) upon the curve; that is, using Fig. 115, we are to determine the length of the parabolic arc OA. It should be appreciated at the outset that such a problem is different in character from any that have occurred in the preceding chapters, or from any that can be solved by elementary mathematics. The solution by integration, however, is readily obtained as follows:

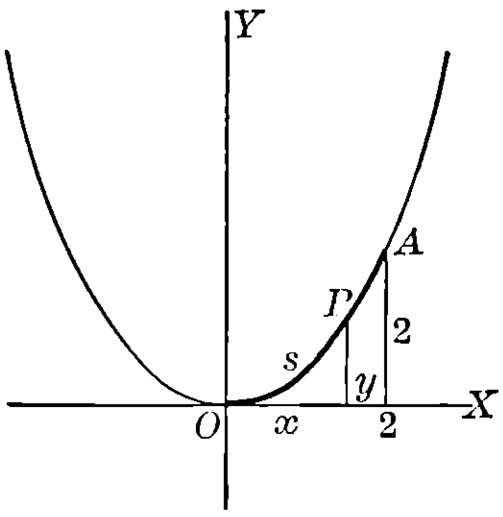

Fig. 115

We first observe that, in accordance with the lemma of Art. 49, the distance s along the curve from the fixed point O to a variable point $P = (x, y)$ is such a function of x that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{1}$$

In the present instance, $y = \frac{1}{2}x^2$ so that $\frac{dy}{dx} = x$, thus giving (1) the form

$$\frac{ds}{dx} = \sqrt{1 + x^2}$$

or

$$ds = \sqrt{1 + x^2}\,dx.$$

Therefore, $$s = \int \sqrt{1 + x^2}\, dx.$$

This integral, which may be found by an integration by parts or by the substitution $x = \tan z$, or may now be taken directly from integral tables, has already been obtained in Ex. 19, page 240. Thus, we have

$$s = \tfrac{1}{2} x\sqrt{1 + x^2} + \tfrac{1}{2} \log (x + \sqrt{1 + x^2}) + C. \qquad (2)$$

Thus far we have made no explicit use of the fact that, in the present example, s is to be measured from the *origin;* that is, we are to have $s = 0$ when $x = 0$. This condition, when introduced into (2), completely determines the constant C to be used in the present instance. Thus, by placing $s = 0$ and $x = 0$ in (2) we see that C must have such a value that

$$0 = \tfrac{1}{2} \cdot 0\sqrt{1 + 0} + \tfrac{1}{2} \log (0 + \sqrt{1 + 0}) + C$$

or $$0 = 0 + 0 + C,$$

from which we obtain $C = 0$.

Equation (2) thus takes the following definite form:

$$s = \tfrac{1}{2}[x\sqrt{1 + x^2} + \log (x + \sqrt{1 + x^2})]. \qquad (3)$$

This equation determines the distance s along the curve from the origin to *any* point whose abscissa is x. What we wish in the present example is to determine s as measured from the origin to the *particular* point A having the abscissa $x = 2$. It remains, then, but to put $x = 2$ in (3) to get the final answer, the result being

$$\tfrac{1}{2}[2\sqrt{5} + \log (2 + \sqrt{5})].$$

129. Generalization. A careful examination of the details in Art. 128 shows that the processes there employed are applicable to finding the length of arc along any curve between any two points upon it. We need have given only the equation of the curve and the coördinates of the two points between which the distance is to be determined. The only difficulty lies in our ability to find, either directly or from tables, the integral involved, and this, indeed, is not always possible for the reasons stated in Art. 107. The process of finding the length of a curve is commonly called **rectification.**

In case the curve is defined by parametric equations instead of by an equation of the simple x, y form occurring in Art. 128, the process is to be carried out in the manner illustrated in the following example.

Example. Find the entire length of one arch of the cycloid.

SOLUTION. Referring to the footnote on page 101, the equations of the cycloid in parametric form are

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta), \tag{1}$$

θ being the angle MCP (Fig. 65) at the center of the rolling circle, this angle varying from $\theta = 0$ to $\theta = 2\pi$ as the complete arch OVD is described by P.

From (1) we have

$$dx = a(1 - \cos\theta)\,d\theta, \qquad dy = a\sin\theta\,d\theta. \tag{2}$$

Hence the lemma of Art. 49 gives in the present instance

$$\frac{ds}{a(1-\cos\theta)\,d\theta} = \sqrt{1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}} = \sqrt{\frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2}}$$

$$= \frac{\sqrt{2(1-\cos\theta)}}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}}{1-\cos\theta}.$$

This relation, when solved for ds, gives

$$ds = 2a\sin\frac{\theta}{2}\,d\theta.$$

Therefore,

$$s = 2a\int\sin\frac{\theta}{2}\,d\theta = 4a\int\sin\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right) = -4a\cos\frac{\theta}{2} + C. \tag{3}$$

Since $s = 0$ when $\theta = 0$, we have

$$0 = -4a\cos 0 + C = -4a + C,$$

so that $C = 4a$.

Relation (3) thus assumes the definite form

$$s = 4a\left(1 - \cos\frac{\theta}{2}\right). \tag{4}$$

Wishing to obtain the length, L, of the entire arch, it remains but to place $\theta = 2\pi$ in (4), thus giving as the desired result

$$L = 4a(1 - \cos\pi) = 8a.$$

The interesting fact thus appears that the length of one complete arch is just eight times the radius of the rolling circle.

EXERCISES

1. Find the length of the parabola $y = \frac{1}{2}x^2$ from the vertex to the point (4, 8). *Ans.* $\frac{1}{2}[4\sqrt{17} + \log(4 + \sqrt{17})]$.

HINT. See Art. 128.

2. Find the length of the parabola $y = 4x^2$ from the vertex to the point (1, 4). *Ans.* $\frac{1}{16}[8\sqrt{65} + \log(8 + \sqrt{65})]$.

3. Find the length of the parabola $y^2 = 4x$ from the vertex to the point (1, 2). Draw a figure. *Ans.* $\sqrt{2} + \log(1 + \sqrt{2})$.

4. Find the length of the arc of the upper branch of the semicubical parabola $9\,y^2 = 4\,x^3$ extending from the origin to the point on the curve having the abscissa $x = 3$. See diagram, page 368. *Ans.* $\frac{14}{3}$.

HINT. With the origin as starting point, we find $C = -\frac{8}{27}$. Compare Art. 128.

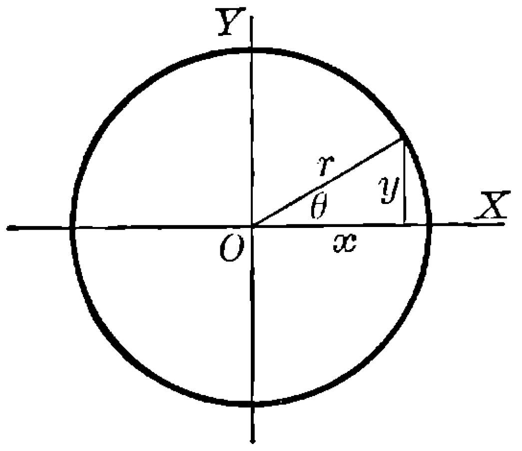

FIG. 116

5. Show by the method of Art. 128 that the circumference of the circle $x^2 + y^2 = r^2$ is of length $2\,\pi r$. Do the same also by the method of Art. 129, the parametric equations of the circle being taken as

$$x = r\cos\theta,\quad y = r\sin\theta.$$

HINT. Confine the attention to one quadrant only, subsequently multiplying the result by 4.

6. Find the length of the arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from its vertex $(0, a)$ to the point (x_1, y_1) on the curve. Moreover, from your result show that, if s_1 represents the arc length in question, we have $s_1^2 = y_1^2 - a^2$, which is a characteristic property of the catenary. See page 368. *Ans.* $= \frac{a}{2}\left(e^{\frac{x_1}{a}} - e^{-\frac{x_1}{a}}\right)$.

7. Prove that if s is the arc length along the cycloid considered in Art. 129, the starting point being as there taken, then the length s extending to any point (x, y) on the curve will be such that $s^2 = 8\,a(s - y)$. This is a characteristic property of the cycloid. Interpret geometrically.

8. Prove that the entire length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $6\,a$. For figure, see page 369.

HINT. $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$. Hence $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

9. The equation of a cardioid may be written in parametric form as follows:

$$x = a(2\cos\phi + \cos 2\,\phi),\quad y = a(2\sin\phi + \sin 2\,\phi).$$

Show that $ds = 4\,a\cos\left(\frac{\phi}{2}\right)d\phi$ and that the total length of the curve is $16\,a$.

130. Area under a curve. As a second geometrical application of the process of integration we shall consider the general problem of determining the area bounded by a given curve $y = f(x)$, the X-axis and any two ordinates drawn to the curve, as illustrated by the shaded area in Fig. 117. For this purpose we shall first need to establish the following lemma, this lemma playing much the same rôle in the present problem as the lemma of Art. 49 played in determining length of arc (Art. 128).

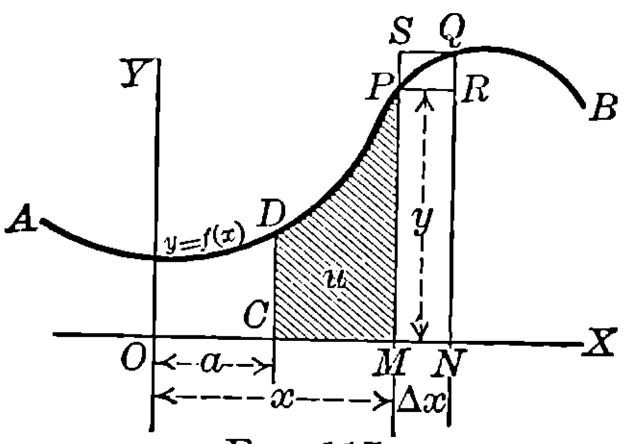

FIG. 117

Lemma. *Let u be the area bounded by the (continuous) curve $y = f(x)$, the X-axis, the fixed ordinate CD drawn at $x = a$ and the variable ordinate MP drawn at $x = x$. Then u will be such a function of x that*

$$\frac{du}{dx} = f(x)$$

or

$$du = f(x)\, dx.$$

PROOF. Let x take on a small increment $MN = \Delta x$. Then u takes on a corresponding increment $\Delta u = area\ MNQP$ By drawing the lines PR, PS, SQ so as to form the rectangles $MNRP$, $MNQS$, we see that

$$area\ MNRP < area\ MNQP < area\ MNQS,$$

or

$$MP\ \ \Delta x < \Delta u < NQ \cdot \Delta x.$$

Dividing by Δx, we thus have

$$MP < \frac{\Delta u}{\Delta x} < NQ.$$

Now let Δx approach zero. As this is done MP remains fixed while NQ approaches MP as its limit. At the same time $\frac{\Delta u}{\Delta x}$ approaches as its limit the derivative $\frac{du}{dx}$. Thus we have

$$\frac{du}{dx} = MP = y,$$

or

$$du = y\, dx = f(x)\, dx,$$

as was to be proved.

The applications of this lemma are immediate as illustrated below:

Example. Find the area bounded by the parabola $y = x^2$, the X-axis and the ordinates corresponding to $x = 1$ and $x = 2$.

SOLUTION. We here have $y = x^2$, so that the foregoing lemma gives

$$du = x^2\, dx,$$

where u represents the (variable) area extending to the ordinate drawn at the (variable) distance x from O. It follows that

$$u = \int x^2\, dx = \frac{x^3}{3} + C. \qquad (1)$$

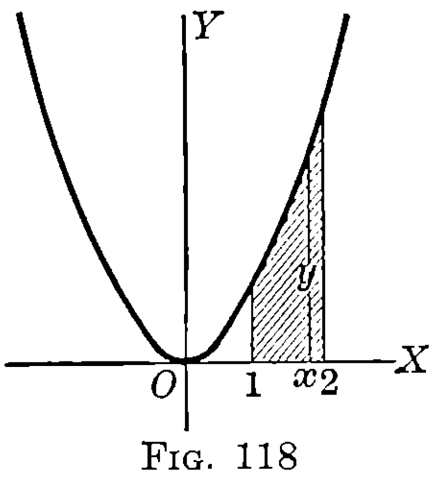

FIG. 118

In order to determine the value of the constant C which should be used in the present instance, we now observe that, by hypothesis, $u = 0$ when $x = 1$. Thus, (1) gives

$$0 = \frac{1^3}{3} + C = \frac{1}{3} + C \quad \text{and therefore} \quad C = -\frac{1}{3}.$$

Relation (1) therefore takes the precise form

$$u = \tfrac{1}{3}(x^3 - 1). \tag{2}$$

Wishing now to find the area extending to the particular distance $x = 2$, we have but to place $x = 2$ in (2). The desired area A is therefore

$$A = \tfrac{1}{3}(2^3 - 1) = \tfrac{7}{3} = 2\tfrac{1}{3}.$$

The area thus obtained is called the area *under* the curve $y = x^2$ between $x = 1$ and $x = 2$.

Note. In applying the foregoing process a negative result will ensue in case the curve lies entirely below the X-axis instead of, as in Figs. 117 and 118, entirely above it. That this should be the case is readily seen upon carrying through the proof of the lemma for such a case. The *actual* area is then obtained by merely taking the *numerical value* of the negative result obtained. In case the curve lies partly above and partly below the X-axis, the process leads to a result whose numerical value is equal to the difference between the areas existing on the positive and negative sides of the X-axis.

EXERCISES

1. Find the area under the curve $y = x^2$ between $x = 2$ and $x = 3$. *Ans.* $6\frac{1}{3}$.

2. Find the area under the curve $y = x^3$ and lying between the origin and the line $x = 4$. *Ans.* 64.

3. Show that the area of one arch of the curve $y = \sin x$ (sinusoid) is equal to 2. See page 370.

4. Find the area between the upper branch of the curve $y^2 = ax^3$ (semicubical parabola), the X-axis, and the line $x = a$. For a graph of this curve see page 368. *Ans.* $\frac{2}{5}a^3$.

5. Find the area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. See page 368. *Ans.* $\frac{1}{6}a^2$.

6. Find the area lying between the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$. Draw a figure. *Ans.* $\frac{16}{3}a^2$.

7. Show by integration that the area of a circle of radius r is πr^2.

8. Find the area of one arch of the cycloid whose parametric equations are

$$x = a\,(\theta - \sin\theta), \quad y = a\,(1 - \cos\theta). \qquad \textit{Ans. } 3\,\pi a^2.$$

HINT. See Example in Art. 129.

131. Applications to physics. The process of integration has manifold applications in connection with physical problems. We shall merely give at this point certain illustrations of this connection, delaying the further development of the subject to later chapters.

Example 1. A particle P starts from rest at a certain point O and moves along a straight line OA in such a manner as to have a constant acceleration a. Determine the formulas expressing the velocity v of the particle and its distance s from O as functions of the time t.

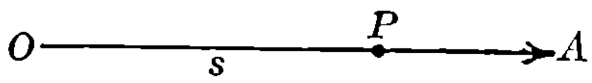

FIG. 119

SOLUTION. From formula (6) of Art. 54 we here have

$$\frac{dv}{dt} = a, \quad \text{or} \quad dv = a\,dt.$$

Hence,

$$v = \int a\,dt = at + C. \tag{1}$$

But $v = 0$ when $t = 0$. Introducing this condition in (1), we obtain $0 = 0 + C$ so that $C = 0$. The desired formula for v in terms of t is therefore

$$v = at. \tag{2}$$

Recalling now that $\frac{ds}{dt} = v$ (formula (2), Art. 54), (2) takes the form

$$\frac{ds}{dt} = at, \quad \text{or} \quad ds = at\,dt.$$

Hence,

$$s = \int at\,dt = a\int t\,dt = \frac{at^2}{2} + C. \tag{3}$$

But $s = 0$ when $t = 0$. Introducing this condition in (3), we obtain $C = 0$. The desired formula for s in terms of t is therefore

$$s = \tfrac{1}{2}\,at^2. \tag{4}$$

It is to be observed that if in particular $a = g = 32.2$, formulas (2) and (4) become, as they should, the familiar "laws of a falling body" considered in physics. See also Ex. 3, page 109.

Example 2. A ball is thrown (projected) horizontally with a velocity of a feet per second from a point O which is h feet from the ground. Discuss the path of its descent and find the velocity with which the ball strikes the ground (assumed level).

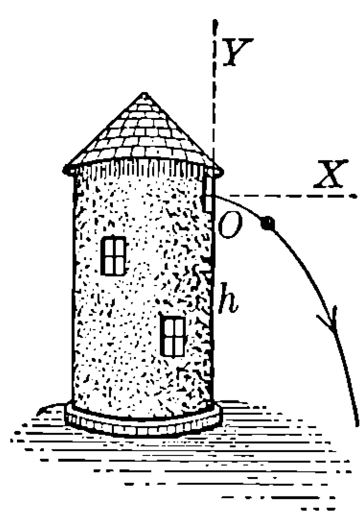

FIG. 120

SOLUTION. Take as positive X-axis the horizontal line OX along which the ball is thrown, and let the positive Y-axis be taken vertically upward at O. Then, inasmuch as the ball is thrown horizontally, the horizontal component v_x of its velocity will remain the same (resistance of the air being neglected) at all points of the descent, being as at the beginning. That is, we shall have

$$v_x = a. \tag{1}$$

In order to obtain a corresponding formula for the vertical component v_y of the velocity we note that, inasmuch as the ball is constantly urged by

gravity in the direction of the negative Y-axis, its vertical acceleration will be the negative constant $-g = -32.2$ feet per (second)^2. Hence, we have (Art. 53)

$$\frac{dv_y}{dt} = -g, \quad \text{or} \quad dv_y = -g\,dt,$$

from which it follows that

$$v_y = -g\int dt = -gt + C.$$

But $v_y = 0$ when $t = 0$. Hence $C = 0$ so that the last equation takes the precise form

$$v_y = -gt. \tag{2}$$

Equations (1) and (2), which in themselves determine completely the X- and Y-components of the velocity at any time t during the motion of the ball, are respectively equivalent, by Art. 53, to the following:

$$\frac{dx}{dt} = a, \qquad \frac{dy}{dt} = -gt,$$

or

$$dx = a\,dt, \qquad dy = -gt\,dt.$$

Integrating, we have

$$x = at + C_1, \qquad y = -g\frac{t^2}{2} + C_2,$$

where C_1 and C_2 are constants. To find their values we note that $x = 0$ when $t = 0$, and $y = 0$ when $t = 0$. Thus we have directly $C_1 = 0$, $C_2 = 0$. We therefore have at any time t during the motion

$$x = at, \qquad y = -\frac{gt^2}{2}. \tag{3}$$

Equations (3) completely determine the position (x, y) of the ball at any time t. These equations are, in fact, the parametric equations of the path, t being the parameter. In order to find the equation of the path in the usual x, y-form, we have only to eliminate t from (3). Thus, the first equation in (3) gives $t = \frac{x}{a}$ and with this expression used for t the second equation becomes

$$y = -\frac{g}{2a^2}x^2. \tag{4}$$

This is at once recognized as the equation of a parabola, being in fact of the standard type of a parabola which opens in the direction of the negative Y-axis. The position of its focus and other important properties may, if desired, be now obtained by analytic geometry.

Thus, the first question asked in the example is answered. As to the second question, we have but to return to (1) and (2) together with formula (1) of Art. 56 to see that the velocity v of the ball at any time t during the motion is determined by the formula

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{a^2 + g^2t^2}. \tag{5}$$

Now, at the instant in question, namely, when the ball strikes the ground, we have $y = -h$. Placing $y = -h$ in the second of equations (3), we obtain

$$-h = -\frac{gt^2}{2},$$

from which, solving for t, it appears that the ball strikes the ground when

$$t = \sqrt{\frac{2h}{g}}.$$

With this particular value for t used in (5), we obtain as the desired velocity of striking the ground

$$V = \sqrt{a^2 + 2gh}. \tag{6}$$

Thus (4) and (6) answer completely both the questions occurring in the Example.

EXERCISES

1. A particle P starts from rest at a certain point O and moves along a straight line OA in such a manner that its acceleration varies directly as the time elapsed since starting; that is, the acceleration is equal to kt, where k is a constant. Determine the formulas expressing the velocity v and the distance s of P from O as functions of t. Compare Example 1, Art. 131. *Ans.* $v = \dfrac{kt^2}{2}$, $s = \dfrac{kt^3}{6}$.

2. Work Ex. 1 in case the particle has an initial velocity of v_0 instead of starting from rest, other conditions remaining the same.

Ans. $v = \dfrac{kt^2}{2} + v_0$, $s = \dfrac{kt^3}{6} + v_0t$.

3. A particle starts at the origin and travels along a plane curve in such a way that the X- and Y-components of its velocity at any time t are determined by the equations

$$v_x = t, \qquad v_y = t^3.$$

Find the equation of the path. *Ans.* $y = x^2$.

4. Work Ex. 3 in case the X- and Y-components of the acceleration (instead of the velocity) are given by the equations

$$j_x = t, \qquad j_y = t^3,$$

it being assumed also that the particle starts from rest. *Ans.* $x^5 = \frac{250}{243}y^3$.

5. A particle starts from the point $(a, 0)$, a being any constant, and travels in such a manner that the X- and Y-components of its velocity at any time t are given by the equations

$$v_x = -a \sin t, \qquad v_y = a \cos t.$$

Show that the path of the particle will be a circle with center at the origin.

6. If in Ex. 5 the particle starts from the point (a, b) where $b \neq 0$, other conditions remaining the same, show that the path is still a circle. How do its center and radius compare with those pertaining to Ex. 5?

7. Generalize the result obtained in illustrative Example 2, Art. 131 by proving that if a particle is projected with an initial velocity of v_0 inclined at an angle α to the horizon, its path will be a parabola.

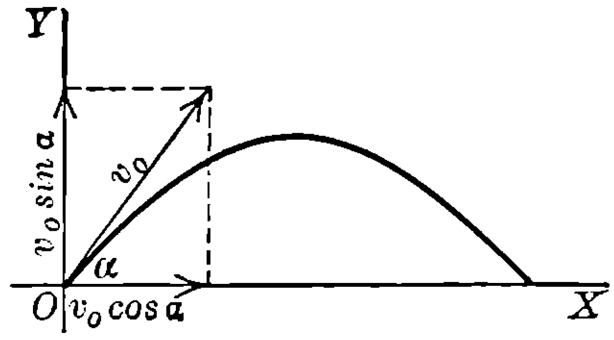

FIG. 121

HINT. Take axes as indicated in Fig. 121 and observe that, under the given conditions, the initial values of the X- and Y-components of the velocity will be respectively $v_0 \cos \alpha$ and $v_0 \sin \alpha$. Show that with axes thus chosen the actual equation of the path becomes

$$y = x \tan \alpha - \frac{gx^2}{2 v_0^2 \cos^2 \alpha}.$$

132. Miscellaneous applications. Besides the applications of integration already noted many others follow directly from the various formulas obtained in Chapters II–V

Example 1. Determine the equation of the curve (or curves) such that the slope at any point upon it is equal to the ordinate of the point.

SOLUTION. The given condition is expressed by the equation

$$\frac{dy}{dx} = y, \quad \text{or} \quad \frac{dy}{y} = dx.$$

Integrating, using formula [5], Art. 108, we obtain

$$\log y = x + C; \quad C = \text{arbitrary constant};$$

and this is equivalent to

$$y = Ce^x; \quad C = \text{arbitrary constant}, \qquad (1)$$

as appears by taking the logarithm of each member of the latter equation, noting that

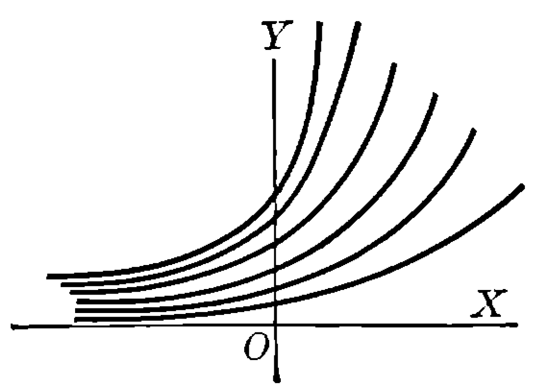

FIG. 122

$$\log Ce^x = \log C + \log e^x = \log C + x = x + \log C$$

and that, inasmuch as C is an *arbitrary* constant, $\log C$ may here be replaced by a single letter such as C to represent an arbitrary constant.

Whatever the value assigned to C, the curve (1) is a so-called "exponential curve" (see page 370) passing through the point $(0, C)$. By assigning different values to C we thus obtain a *family* of curves, as indicated in Fig. 122, each member of which has the property specified in the Example at each of its points. Compare Fig. 114, page 212.

Example 2. Determine the equation of the curve (or curves) whose subnormal is constant, being equal to a.

SOLUTION. By Art. 46, the given condition is expressed by the equation

$$y\frac{dy}{dx} = a.$$

Writing this equation in the form

$$y\,dy = a\,dx$$

and integrating both members, we obtain

$$\frac{y^2}{2} = ax + C, \quad \text{or} \quad y^2 = 2\,ax + 2\,C,$$

which, recalling that C is an arbitrary constant, is equivalent to

$$y^2 = 2\,ax + C. \tag{1}$$

The condition proposed in the Example is thus satisfied by (and only by) the parabolas of the family (1). The student is advised to draw a figure to illustrate the meaning of this result.

EXERCISES

1. Determine the equation of the family of curves each of which has its slope at any point equal to the abscissa of the point. Draw a figure to illustrate the meaning of your answer. *Ans.* $x^2 = 2\,y + C.$

2. Show that if a curve has its slope at each point equal to the ratio of the abscissa of the point to its ordinate, it belongs to the family of equilateral hyperbolas whose equation is $x^2 - y^2 = C.$

3. Work Ex. 2 in case the negative ratio of the ordinate to the abscissa is used instead of the ratio of the abscissa to the ordinate. *Ans.* $xy = C.$

4. Determine the equation of the family of curves all of which have a constant subtangent, equal to a. See Art. 46. *Ans.* $a \log y = x + C.$

5. Determine the equation of the family of curves all of which have a subnormal at each point equal to the abscissa of the point. *Ans.* $x^2 - y^2 = C.$

6. Show that any curve whose polar subtangent is constant is a reciprocal spiral. See Corollary 1, Art. 101.

7. Given the family of parabolas $x^2 = 2\,y + C$ (see Ex. 1). Show that the equation of the family of curves each of which meets every curve of the first family at right angles is $y + \log x = C$, or $xe^y = C$. Draw a diagram to show the geometric meaning of this result. Two families of curves related in this manner are said to be **orthogonal families**.

8. Determine the family of curves orthogonal to the family of circles $x^2 + y^2 = C$. Draw a figure.

CHAPTER XIV

THE DEFINITE INTEGRAL

133. Definitions. The two values of the integral of $f(x)$ corresponding to $x = a$ and $x = b$ may be denoted respectively by

$$\left[\int f(x)\,dx\right]_{x=a} + C \text{ and } \left[\int f(x)\,dx\right]_{x=b} + C. \tag{1}$$

Here the constant C, although arbitrary, has the same value in each case. The result obtained by subtracting the first of the values (1) from the second we shall represent by the symbol

$$\int_a^b f(x)\,dx$$

so that

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_{x=b} - \left[\int f(x)\,dx\right]_{x=a}$$

or, as it is usually written,

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_{x=a}^{x=b}. \tag{2}$$

This expression is known as the **definite integral** of $f(x)$ *between the limits a and b*, or, more briefly, *the definite integral of $f(x)$ from a to b.* Because of their positions, a is termed the **lower limit** of the definite integral, and b the **upper limit.** The interval from a to b is called the **interval of integration.**

Hereafter we shall speak of the former symbol $\int f(x)\,dx$ as the **indefinite integral** of $f(x)$ in order to distinguish it from the definite integral as defined above. Observe that the indefinite integral of $f(x)$ is a function of x which always involves, as indicated before, an arbitrary constant C, while the definite integral of $f(x)$ is not a function of x at all, but is a fixed numerical value. Hence the appropriateness of the terms "definite" and "indefinite" as employed above.

Thus, $\int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_{x=2}^{x=3} = \left[\frac{x^3}{3}\right]_{x=3} - \left[\frac{x^3}{3}\right]_{x=2} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}.$

Similarly, $\int_0^{\frac{\pi}{2}} \sin x\,dx = \left[-\cos x\right]_{x=0}^{x=\frac{\pi}{2}} = \left[-\cos x\right]_{x=\frac{\pi}{2}} - \left[-\cos x\right]_{x=0}$

$= 0 - (-1) = 1.$

From the definition (2) it follows that if the limits a and b are interchanged, the effect upon the definite integral is merely to change its sign. That is, we may always write

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{3}$$

EXERCISES

Evaluate each of the following definite integrals:

1. $\int_0^1 x^3\,dx.$ *Ans.* $\frac{1}{4}$.
2. $\int_2^3 \frac{x\,dx}{1+x^2}.$ *Ans.* $\log\sqrt{2}$.
3. $\int_{-1}^0 \frac{dx}{1+x^2}.$ *Ans.* $\frac{\pi}{4}$.
4. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2-x^2}}.$ *Ans.* $\frac{\pi}{6}$.
5. $\int_0^\pi \cos\phi\,d\phi.$ *Ans.* 0.
6. $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx.$ *Ans.* $\frac{\pi}{4}$.
7. $\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta.$ *Ans.* $\frac{\pi}{4}$.
8. $\int_2^3 \frac{x^2\,dx}{x+1}.$ *Ans.* $\frac{3}{2} + \log\frac{4}{3}$.
9. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x\,dx}{1+\sin^2 x}.$ *Ans.* $\frac{\pi}{2}$.
10. $\int_0^\pi x\cos x\,dx.$ *Ans.* -2.
11. $\int_1^2 \log x\,dx.$ *Ans.* $2\log 2 - 1$.
12. $\int_0^1 xe^x\,dx.$ *Ans.* 1.
13. $\int_0^\pi \cos^3 x\,dx.$ *Ans.* 0.
14. $\int_0^r \frac{r\,dx}{\sqrt{r^2+x^2}}.$ *Ans.* $r\log(1+\sqrt{2})$.
15. $\int_0^1 \operatorname{arc\,sin} x\,dx.$ *Ans.* $\frac{\pi}{2} - 1$.
16. $\int_0^1 \operatorname{arc\,tan} x\,dx.$
17. $\int_0^1 x\log x\,dx.$ *Ans.* $-\frac{1}{4}$.
18. $\int_0^1 x^2\log x\,dx.$ *Ans.* $-\frac{1}{9}$.
19. $\int_0^{\frac{\pi}{2}} \phi^2\sin\phi\,d\phi.$ *Ans.* $\pi - 2$.
20. Evaluate each of the following definite integrals:

(*a*) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$; (*c*) $\int_1^e x\log x\,dx$; (*e*) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\,dx$;

(*b*) $\int_3^4 \frac{t\,dt}{1+t^2}$; (*d*) $\int_0^1 x^2 e^x\,dx$; (*f*) $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$.

134. Change of limits corresponding to change of variable. If, in evaluating a definite integral, a substitution of a new variable has been introduced, as illustrated in Arts. 124–125, it is at no time necessary to return to the original variable in order to secure the result desired. All that is necessary is to alter the limits of integration on the transformed integral, doing so in a way consistent with the substitution introduced.

Example. Find $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2 - x^2}}$, making the substitution $x = a \sin \theta$.

SOLUTION. $\sqrt{a^2 - x^2} = a\sqrt{1 - \sin^2 \theta} = a \cos \theta, \quad dx = a \cos \theta \, d\theta.$

Also, when $x = \frac{1}{2} a$ we have $\frac{1}{2} a = a \sin \theta$, hence $\sin \theta = \frac{1}{2}$ and therefore

$$\theta = \frac{\pi}{6}.$$

Again, when $x = 0$ we have $0 = a \sin \theta$, hence $\theta = 0$.

Hence,
$$\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{a \cos \theta}{a \cos \theta} d\theta = \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}.$$

EXERCISES

1. Establish each of the following, using the substitution indicated:

(a) $\int_0^1 \frac{x^2\, dx}{(x+1)^2} = \frac{3}{2} - 2 \log 2$; Substitute $x + 1 = z$.

(b) $\int_3^8 \frac{dx}{x\sqrt{x+1}} = \log \frac{3}{2}$; Substitute $\sqrt{x+1} = z$.

(c) $\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{1}{4} \pi a^2$; Substitute $x = a \sin \theta$.

(d) $\int_1^{\sqrt{3}} \frac{dx}{x(1+x^2)} = \log \frac{\sqrt{6}}{2}$; Substitute $x = \tan \theta$.

(e) $\int_0^{\frac{\pi}{2}} \frac{dt}{3 + 2\cos t} = \frac{2}{\sqrt{5}} \arctan \frac{1}{\sqrt{5}}$; Substitute $z = \tan \frac{t}{2}$.

2. Establish each of the following, using a suitable substitution of variable in each:

(a) $\int_1^6 \frac{x\, dx}{\sqrt{3+x}} = \frac{20}{3}$;

(b) $\int_0^4 \frac{dx}{1+\sqrt{x}} = 4 - 2\log 3$;

(c) $\int_1^2 \frac{dx}{x^2\sqrt{1+x^2}} = \sqrt{2} - \frac{1}{2}\sqrt{5}$;

(d) $\int_0^a y^2\sqrt{a^2 - y^2}\, dy = \frac{\pi a^4}{16}$;

(e) $\int_0^{\log 2} \sqrt{e^x - 1}\, dx = \frac{4-\pi}{2}$;

(f) $\int_{\frac{a}{2}}^a \frac{\sqrt{a^2 - x^2}}{x^2} dx = \sqrt{3} - \frac{\pi}{3}$.

135. Length and area expressed as definite integrals. Returning to the discussions in Arts. 128–130 concerning arc-length and area, we may now establish the following general theorems through which these magnitudes become expressible as definite integrals:

Theorem I. *The arc-length along the curve $y = f(x)$ between the points whose abscissas are respectively $x = a$ and $x = b$ is given by the formula*

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

PROOF. Reasoning as in Arts. 128–129, the arc-length as measured from any fixed point P_0 on the curve to the variable point $P = (x, y)$ is

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

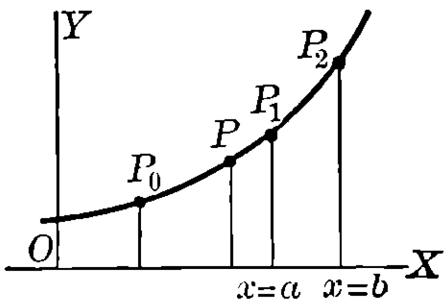

FIG. 123

In particular, then, the values of s from P_0 to the points P_1, P_2 having the abscissas $x = a$, $x = b$ are respectively

$$s_1 = \left[\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\right]_{x=}\qquad s_2 = \left[\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\right]_{x=b}$$

The arc-length from P_1 to P_2 is therefore

$$s = s_2 - s_1 = \left[\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\right]_{x=b} - \left[\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\right]_{x=a}$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The proof is thus complete.

Theorem II. *The area bounded by the curve $y = f(x)$, the X-axis, and the ordinates erected at $x = a$ and $x = b$ is given by the formula*

$$A = \int_a^b f(x)\, dx = \int_a^b y\, dx.$$

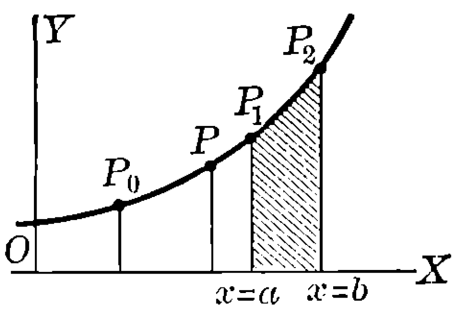

FIG. 124

The proof, which follows directly from the lemma of Art. 130 and is otherwise similar to the foregoing proof for Theorem I, is left to the student as an exercise.

It is to be observed that, by the use of the foregoing theorems, the determination of arc-lengths and areas becomes much simplified (compare Arts. 128–130) since these theorems avoid all reference to the constant C of integration.

Example. Find the area bounded by the parabola $y = x^2$, the X-axis, and the ordinates drawn at $x = 1$ and $x = 2$.

SOLUTION. (Compare Example, Art. 130.) We have at once, by Theorem II,

$$A = \int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_{x=1}^{x=2} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2\tfrac{1}{3}.$$

EXERCISES

By use of Theorems I and II work the following:

1. Exs. 1–4, pages 247–248. **2.** Exs. 1–6, page 250.

3. Exs. 6–8, page 248. **4.** Ex. 7, page 250.

5. Prove that the area bounded by a parabola and any chord perpendicular to the axis is equal to two thirds of the rectangle circumscribing this area.

6. Find the area bounded by the curve $y = \frac{\log x}{x}$, the X-axis, and the maximum ordinate. *Ans.* $\frac{1}{2}$.

136. Improper integrals. Suppose that in the definite integral

$$\int_a^b f(x)\,dx \tag{1}$$

the upper limit b is allowed to increase indefinitely, that is, suppose that $b \rightarrow +\infty$, the lower limit a meanwhile remaining fixed. The various values which the integral then takes may approach some definite limiting value. If so, this limiting value is called "the integral of $f(x)$ from a to $+\infty$" and is represented by the symbol

$$\int_a^\infty f(x)\,dx. \tag{2}$$

If, on the other hand, the values taken by (1) as $b \rightarrow \infty$ do *not* approach a limit, no meaning can be attached to the expression (2) and the integral is therefore said not to exist.

Thus the integral of the function $\frac{1}{x^2}$ from 1 to $+\infty$ exists and is equal to 1 since we have

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\rightarrow+\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\rightarrow+\infty} \left\{ \left[-\frac{1}{x}\right]_{x=b} - \left[-\frac{1}{x}\right]_{x=1} \right\}$$

$$= \lim_{b\rightarrow+\infty} \left\{ -\frac{1}{b} + 1 \right\} = 0 + 1 = 1.$$

The geometrical significance of this result appears in Fig. 125. Here the shaded area lying between the curve $y = \frac{1}{x^2}$, the ordinate at $x = 1$, and the positive X-axis approaches the limiting value 1 as this area is allowed to extend indefinitely to the right by taking b larger and larger.

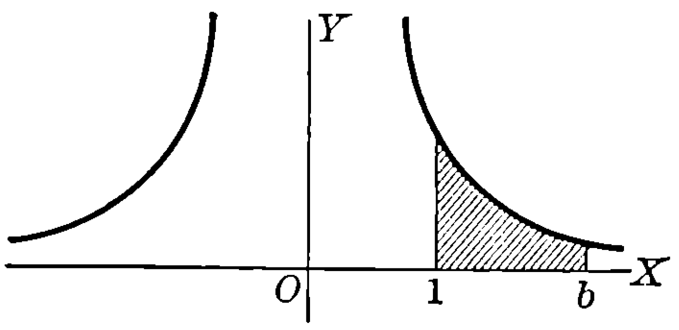

FIG. 125

On the other hand, the integral of the function $\frac{1}{x}$ from 1 to $+\infty$ does not exist since we have

$$\int_1^\infty \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} \left\{ [\log x]_{x=b} - [\log x]_{x=1} \right\}$$
$$= \lim_{b \to +\infty} \left\{ \log b - 0 \right\} = +\infty.$$

Considered geometrically, this result leads to a figure analogous to Fig. 125, but different in that the area between the curve $y = \frac{1}{x}$, the ordinate at $x = 1$, and the positive X-axis increases beyond all limit as this area is extended further and further to the right. The student is advised to draw the figure.

Similarly, "the integral of $f(x)$ from $-\infty$ to b" is defined as follows:

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx. \tag{3}$$

Whenever the limit here indicated on the right exists the integral on the left has a geometrical significance, being the measure of an area which extends indefinitely in the direction of the negative X-axis and has this axis as one of its boundaries.

The integral of $f(x)$ from $-\infty$ to $+\infty$ may now be defined as follows:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx. \tag{4}$$

Here the integral on the left is said to exist only in case the same is true of each of the integrals on the right.

Returning to the original form (1) in which both a and b are regarded as finite, let us now suppose that $f(x)$, instead of remaining finite throughout the interval of integration (a, b) as was assumed in Art. 133, becomes infinite at $x = a$. In such a case the integral (1) is regarded as meaning the following limit provided it exists:

$$\lim_{x \to a^+} \int_x^b f(x)\,dx, \tag{5}$$

where $\lim\limits_{x \to a^+}$ means the limit as x approaches a through values *greater* than a.

For example, the function $\frac{1}{\sqrt{x}}$ becomes infinite at $x = 0$, hence the integral of this function from 0 to 1 furnishes an illustration of the type of integral in question. For this integral we have, in accordance with the definition (5),

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{x \to 0^+} \int_x^1 \frac{dx}{\sqrt{x}} = \lim_{x \to 0^+} \left\{ [2\sqrt{x}]_{x=1} - [2\sqrt{x}]_{x=x} \right\}$$

$$= \lim_{x \to 0^+} \left\{ 2 - 2\sqrt{x} \right\} = 2.$$

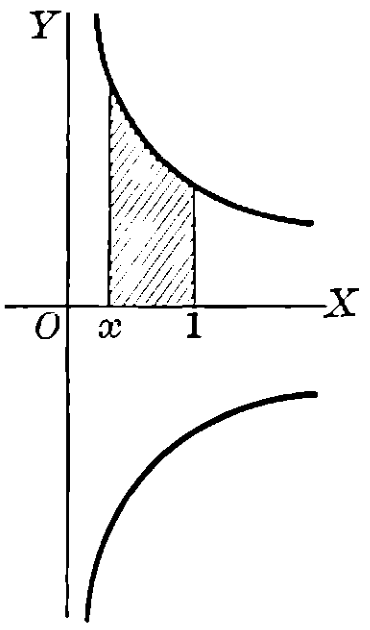

FIG. 126

The geometrical significance of this result appears in Fig. 126, the shaded area approaching the limit 2 as x approaches zero.

Similarly, if $f(x)$ becomes infinite at $x = b$, we define the integral (1) to be the following limit provided it exists:

$$\lim_{x \to b^-} \int_a^x f(x)\,dx, \tag{6}$$

where $\lim\limits_{x \to b^-}$ means the limit as x approaches b through values *less* than b.

Finally, if $f(x)$ becomes infinite at some point $x = p$ lying *between* a and b, we define the integral (1) as follows

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx,$$

where the first of the integrals appearing on the right is to be determined in accordance with (6) and the second in accordance with (5).

Thus,

$$\int_{-1}^1 \frac{dx}{x^{\frac{2}{3}}} = \int_{-1}^0 \frac{dx}{x^{\frac{2}{3}}} + \int_0^1 \frac{dx}{x^{\frac{2}{3}}} = \lim_{x \to 0^-} \int_{-1}^x \frac{dx}{x^{\frac{2}{3}}} + \lim_{x \to 0^+} \int_x^1 \frac{dx}{x^{\frac{2}{3}}}$$

$$= \lim_{x \to 0^-} \left\{ [3x^{\frac{1}{3}}]_{x=x} - [3x^{\frac{1}{3}}]_{x=-1} \right\} + \lim_{x \to 0^+} \left\{ [3x^{\frac{1}{3}}]_{x=1} - [3x^{\frac{1}{3}}]_{x=x} \right\}$$

$$= \lim_{x \to 0^-} \left\{ 3x^{\frac{1}{3}} + 3 \right\} + \lim_{x \to 0^+} \left\{ 3 - 3x^{\frac{1}{3}} \right\} = 3 + 3 = 6.$$

Integrals belonging to any of the types just described are known as **improper integrals** inasmuch as they do not properly belong to the simple type discussed in Art. 133.

EXERCISES

State wherein each of the following is an improper integral and determine its value if it has one. Draw a figure to show the geometric meaning of the result in each of the first six exercises:

1. $\int_1^{\infty}\frac{dx}{x^3}$. *Ans.* $\frac{1}{2}$.

2. $\int_{-\infty}^{-1}\frac{dx}{x^3}$. *Ans.* $-\frac{1}{2}$.

3. $\int_0^{1}\frac{dx}{x^3}$. *Ans.* Non-existent.

4. $\int_{-\infty}^{\infty}\frac{dx}{x^2}$. *Ans.* Non-existent.

5. $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$. *Ans.* π.

6. $\int_0^{\infty}e^{-x}\,dx$. *Ans.* 1.

7. $\int_0^{\infty}\frac{dx}{x+1}$. *Ans.* Non-existent.

8. $\int_0^{\infty}\frac{dx}{a^2+x^2}$. *Ans.* $\frac{\pi}{2a}$

9. $\int_1^{\infty}\frac{dt}{t(1+t)^2}$. *Ans.* $\log 2-\frac{1}{2}$.

10. $\int_0^{\infty}\sin x\,dx$. *Ans.* Non-existent.

11. $\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}$. *Ans.*

12. $\int_0^{1}\log x\,dx$. *Ans.* -1.

13. $\int_0^{1}\frac{dx}{x-1}$. *Ans.* Non-existent.

14. $\int_{-\infty}^{\infty}\frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$. *Ans.* $\frac{2}{a^2}$.

15. $\int_{-a}^{a}\frac{dx}{a^2-x^2}$.

16. $\int_1^{\infty}\frac{dx}{x\sqrt{x^2-1}}$.

17. $\int_0^{2}\frac{x\,dx}{4-x^2}$.

18. $\int_0^{\infty}e^{-x}\cos x\,dx$.

19. By use of definite integrals (proper or improper) determine the following areas:

(*a*) the area under the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ (see Fig. page 368) from $x=-a$ to $x=a$; *Ans.* $a^2\frac{e^2-1}{e}$.

(*b*) the area of the ellipse whose semiaxes are a and b; *Ans.* πab.
HINT. Confine the attention to the first quadrant.

(*c*) the area in the first quadrant under the curve $y=xe^{-\frac{1}{2}x^2}$; *Ans.* 1.

(*d*) the area bounded by the curve $y=xe^x$ and its asymptote; *Ans.* 1.

(*e*) the area bounded by the curve $y=\frac{2x}{1+x^2}$, its asymptote, and the maximum ordinate. *Ans.* 0.693.

20. Employing the parametric equations of the ellipse (see Art. 51), show that the area of the ellipse whose semiaxes are a and b is πab.

CHAPTER XV

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

137. Introduction. We have seen how certain important arc-lengths and areas may be formulated and determined through the process of integration. Many other important magnitudes may likewise be determined by integration, as will be shown in the present chapter. For this purpose it is first desirable to look at the definite integral from an entirely different point of view. This will now be done.

138. The definite integral regarded as the limit of a sum. Let $f(x)$ be any function which is continuous throughout the interval from $x = a$ to $x = b$. Let us divide this interval into n equal parts, each of which we will designate by Δx. At the points of division let us erect ordinates to the curve $y = f(x)$, and finally let us draw the series of rectangles having these ordinates as their altitudes, as indicated in Fig. 127. The sum of the areas of these rectangles evidently forms a good approximation to the area under the curve from $x = a$ to $x = b$, provided the number n of division points is taken very large, in which case Δx would be very small. In fact, the limit of this sum of rectangles as n increases indefinitely, or what is the same thing, as Δx approaches zero, is seen to be the actual area A under the curve. This being clear geometrically, let us now formulate the fact analytically. To do so, we observe that the points of division referred to, counting $x = a$ as the first point, are respectively

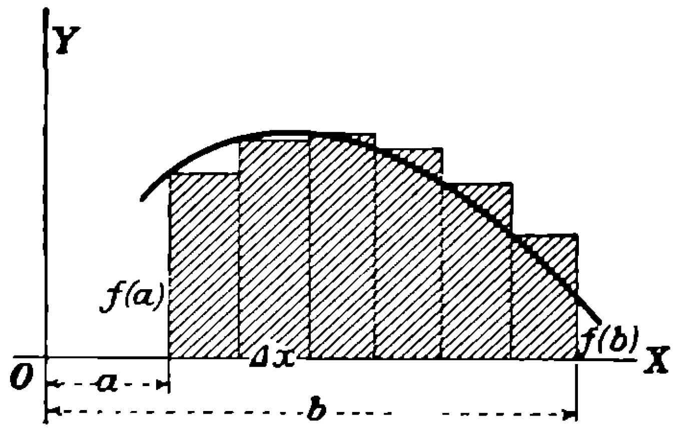

FIG. 127

$$a,\ a + \Delta x,\ a + 2\,\Delta x,\ a + 3\,\Delta x,\ \cdots,\ b - 2\,\Delta x,\ b - \Delta x$$

so that the ordinates at these points have the respective values

$$f(a), f(a + \Delta x), f(a + 2\,\Delta x), f(a + 3\,\Delta x), \quad f(b - 2\,\Delta x), f(b - \Delta x).$$

The area of the first rectangle is thus $f(a) \cdot \Delta x$, the area of the second is $f(a + \Delta x) \cdot \Delta x$, the area of the third is $f(a + 2\,\Delta x) \cdot \Delta x, \cdots$ and the area of the last one is $f(b - \Delta x) \cdot \Delta x$. Hence, their sum is

$$f(a) \cdot \Delta x + f(a + \Delta x) \cdot \Delta x + f(a + 2\,\Delta x) \cdot \Delta x + \cdots + f(b - \Delta x) \cdot \Delta x.$$

The fact which we have just arrived at geometrically thus becomes formulated analytically as follows:

$$A = \lim_{\Delta x \to 0} [f(a)\Delta x + f(a + \Delta x)\Delta x + f(a + 2\,\Delta x)\Delta x + \cdots + f(b - \Delta x)\Delta x].$$

But by Theorem II of Art. 135 A is also expressible as follows:

$$A = \int_a^b f(x)\,dx.$$

Thus, we have two different expressions for A. These two expressions must therefore be equal to each other. Therefore we arrive in summary at the following very important theorem:

Theorem. *Let $f(x)$ be any function which is continuous throughout the interval $a \leqq x \leqq b$. Let this interval be divided into n equal parts each of length Δx and form the sum*

$$f(a)\,\Delta x + f(a + \Delta x)\,\Delta x + f(a + 2\,\Delta x)\,\Delta x + \cdots + f(b - \Delta x)\,\Delta x.$$

If n be now allowed to increase indefinitely, or what is the same thing, if Δx be allowed to approach the limit zero, this sum will approach as its limit a value equal to the definite integral

$$\int_a^b f(x)\,dx.$$

The importance of the theorem lies in the fact that many important magnitudes in geometry and the applied sciences allow themselves to be expressed as the limits of sums of precisely the type referred to in the theorem, as will be shown in the remaining articles of this chapter.

139. Area in polar coördinates. As a first application of the foregoing theorem, we will now determine the formula for area under a curve when polar coördinates instead of rectangular are used, the result being as stated in the following theorem (compare Art. 135)

Theorem. *The area bounded by the curve $\rho = F(\theta)$ and the two radii vectores corresponding to $\theta = \alpha$ and $\theta = \beta$ is given by the formula*

$$A = \frac{1}{2}\int_\alpha^\beta \rho^2\,d\theta.$$

PROOF. Let $\rho_0 = F(\alpha)$. Then, with the pole O as a center, draw − 1 radii vectores $\rho_1, \rho_2, \rho_3, \quad \rho_{n-1}$ separated from each other and from ρ_0 by the common angle $\Delta\theta$ and all lying within the proposed area, as illustrated in Fig. 128. With $\rho_0, \rho_1, \rho_2, \quad \rho_{n-1}$ as radii, describe arcs as shown in the figure and consider the areas of the n circular sectors thus formed. The sum of these areas evidently forms a good approximation to the area sought, provided that n is taken very large, or what is the same thing, in case $\Delta\theta$ is taken very small. Moreover, the limit of this sum as $n \to \infty$, or $\Delta\theta \to 0$, will be the actual area sought. Let us now obtain an expression for this sum.

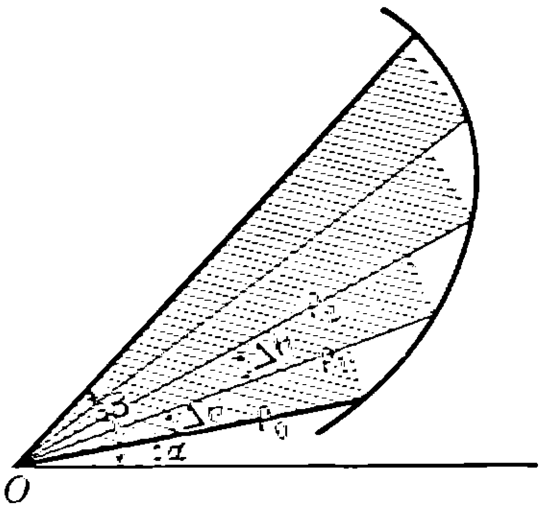

FIG. 128

The radii of the various sectors are respectively

$$\rho_0 = F(\alpha),\ \rho_1 = F(\alpha + \Delta\theta),\ \rho_2 = F(\alpha + 2\,\Delta\theta),\ \cdots,\ \rho_{n-1} = F(\beta - \Delta\theta).$$

Hence, recalling from elementary geometry that the area of a sector of angle ω and radius r is $\frac{1}{2}\,\omega r^2$, the sum of the areas of the inscribed sectors is seen to be

$$\tfrac{1}{2}[F(\alpha)]^2\,\Delta\theta + \tfrac{1}{2}[F(\alpha + \Delta\theta)]^2\,\Delta\theta + \tfrac{1}{2}[F(\alpha + 2\,\Delta\theta)]^2\,\Delta\theta \\ + \quad + \tfrac{1}{2}[F(\beta - \Delta\theta)]^2. \tag{1}$$

But this sum is of the type occurring in the theorem of Art. 138, being made up of the successive values of a certain function, namely, $\frac{1}{2}[F(\theta)]^2$, at the points $\theta = \alpha, \theta = \alpha + \Delta\theta, \theta = \alpha + 2\,\Delta\theta, \cdots, \theta = \beta - \Delta\theta$, each such value being multiplied by $\Delta\theta$. Applying the theorem, we may now say that the desired limit of the sum (1) as $\Delta\theta \to 0$ is given by the formula

$$\tfrac{1}{2}\int_\alpha^\beta [F(\theta)]^2 d\theta = \tfrac{1}{2}\int_\alpha^\beta \rho^2\, d\theta.$$

The proof is thus complete.

Example. Find the area swept out by the radius vector of the spiral of Archimedes $\rho = a\theta$ in the interval from $\theta = 0$ to $\theta = 2\,\pi$.

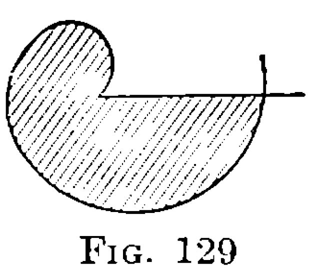

FIG. 129

SOLUTION. Calling A the area in question, we have at once, by the foregoing theorem,

$$A = \frac{1}{2}\int_0^{2\pi} \rho^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{a^2}{2}\left\{\left[\frac{\theta^3}{3}\right]_{\theta=2\pi} - \left[\frac{\theta^3}{3}\right]_{\theta=0}\right\} = \frac{a^2}{2}\cdot\frac{8\,\pi^3}{3} = \frac{4}{3}\pi^3 a^2.$$

EXERCISES

1. Determine the entire area bounded by the curve (circle) $\rho = 2a\cos\theta$. See Fig. 130. *Ans.* πa^2.

2. Determine the entire area bounded by the curve (circle) $\rho = 2a\sin\theta$. Draw a figure. *Ans.* πa^2.

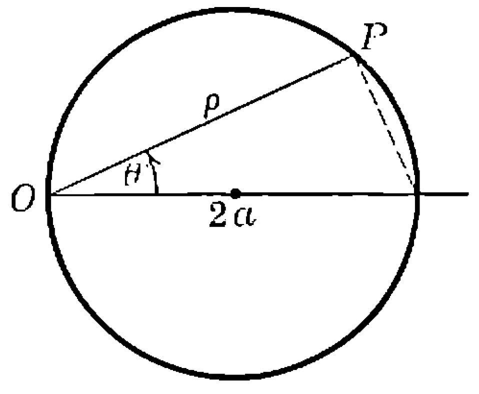

FIG. 130

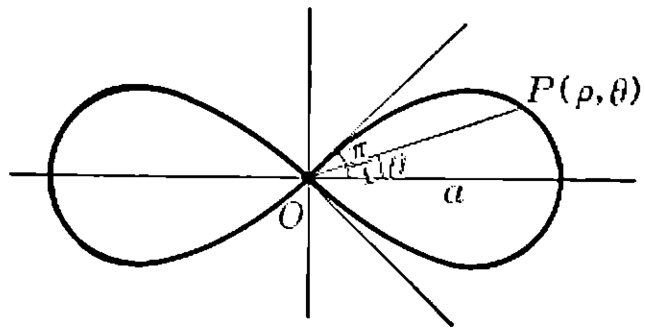

FIG. 131. LEMNISCATE

3. Determine the area bounded by the lemniscate $\rho^2 = a^2\cos 2\theta$. *Ans.* a^2.

4. Determine the area bounded by one loop of the curve (four-leafed rose) $\rho = a\cos 2\theta$. See Fig. 132. *Ans.*

5. Determine the area bounded by the cardioid $\rho = a(1 + \cos\theta)$. *Ans.* $\frac{3}{2}\pi a^2$.

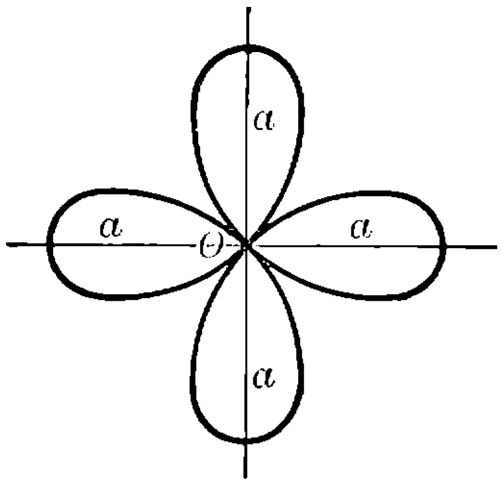

FIG. 132. FOUR-LEAFED ROSE

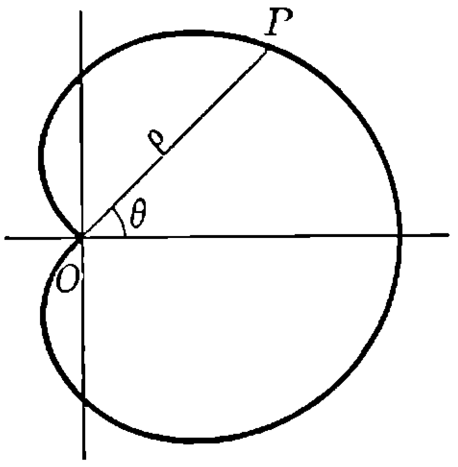

FIG. 133. CARDIOID

6. Work Ex. 4 when the curve $\rho = a\sin 2\theta$ is used. Draw a figure. *Ans.*

7. Prove that the entire area bounded by the curve (three-leafed rose) $\rho = a\sin 3\theta$ is equal to one fourth the area of the circumscribing circle. Draw a figure.

8. Show that the result stated in Ex. 7 holds true also when the curve $\rho = a\cos 3\theta$ is used. Draw a figure.

9. Determine the area bounded by the curve $\rho = a\sin^3\theta$. *Ans.* $\dfrac{5\pi a^2}{32}$.

10. Prove that the area swept over by the radius vector of the spiral $\rho = e^\theta$ as θ varies from $\theta = -\infty$ to any given value is equal to one fourth the area of the square drawn on the radius vector in its final position. Illustrate by diagram.

11. Determine the area bounded by the curve (parabola) $\rho = a\sec^2\dfrac{\theta}{2}$ and the line through the pole perpendicular to the initial line. See page 368. *Ans.* $\frac{8}{3}$

12. Prove that the area bounded by any two radii vectores of the hyperbolic spiral $\rho\theta = a$ is proportional to the difference between the lengths of these radii.

13. Determine each of the following areas:

(*a*) Within the first quadrant and bounded by the curve $\rho^2 = \theta$. Draw a figure.

(*b*) Within the second quadrant and bounded by the curve $\rho = a(1 - \cos\theta)$.

(*c*) Inclosed by the curve $\rho = a \sin\frac{\theta}{2}$.

(*d*) Inclosed by the curve $\rho^2 \cos\theta = a^2 \sin 3\theta$.

140. Volumes of solids of revolution. The volume generated by revolving about the X-axis the area under a given curve $y = f(x)$ may now be formulated as follows:

Theorem. *The volume V generated by revolving about the X-axis the area under the curve $y = f(x)$ between the abscissas $x = a$ and $x = b$ is given by the formula*

$$V = \pi\int_a^b y^2\,dx.$$

PROOF. Returning to Fig. 127, let us suppose that the entire part of the figure lying in the first quadrant is revolved about the X-axis through one complete revolution. As this is done each of the small rectangles generates a right circular cylinder and the sum of these cylinders when their number, n, is large forms a good approximation to the volume generated by the revolution of the entire area beneath the curve. Moreover, the limit of this sum may be regarded as the actual value of this latter volume, which is the volume in question.

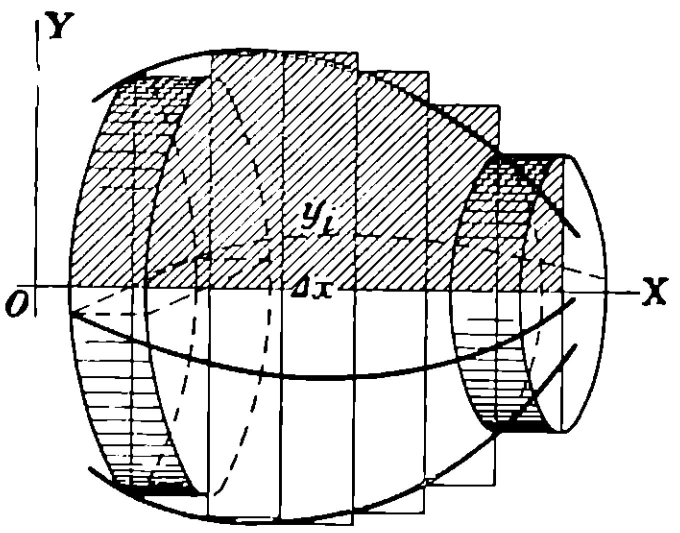

FIG. 134

This being premised, let y_0 represent the ordinate to the given curve at $x = a$ and let $y_1, y_2, \ldots, y_{n-1}$ be the ordinates at the various points of division of the interval (a, b). Then the volume of the cylinder generated by the revolution of the ith rectangle is the area of its base, πy_i^2, multiplied by its altitude, Δx; that is, $\pi y_i^2\,\Delta x$. The sum of such volumes will therefore consist of the sum of these expressions as i ranges from 0 to $n - 1$. But this sum is of the type occurring in the theorem of Art. 138, the function $f(x)$ in the present case being πy^2, and the limit of this sum will be as stated in the theorem.

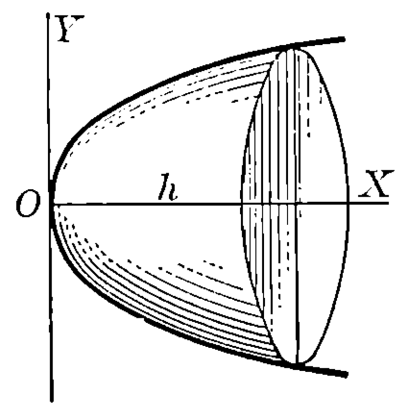

FIG. 135

Example. Find the volume of the solid generated by revolving about the X-axis that portion of the area under the parabola $y^2 = 4\,px$ which lies between the origin and the line $x = h$.

SOLUTION. The foregoing theorem gives at once as the result

$$V = \pi\int_0^h y^2\,dx = \pi\int_0^h 4\,px\,dx = 4\,\pi p\int_0^h x\,dx = 2\,\pi\,ph^2.$$

141. Generalization. If in Art. 140 the revolution, instead of being about the X-axis, is taken about any line parallel to that axis, as $y = k$, it is easily seen that the volume of the solid generated will be given by the formula

$$V = \pi\int_a^b (y - k)^2\,dx. \tag{1}$$

The proof of this fact, being analogous in all respects to that of the theorem in Art. 140, is left to the student. It is to be noted that for the particular case in which $k = 0$ formula (1) reduces, as it should, to that of the preceding article.

Similarly, if the revolution takes place about any line parallel to the Y-axis, as $x = l$, the volume of the solid generated is seen to be given by the formula

$$V = \pi\int_c^d (x - l)^2\,dy \tag{2}$$

where (c, d) is the interval along the Y-axis over which the curve lies.

Example. The area bounded by the parabola $y^2 = 4\,px$ and the line $x = p$ revolves about this line. Find the volume generated.

SOLUTION. We first observe that the revolution here takes place about a line parallel to the Y-axis so that formula (2) applies. In the notation of that formula we here have $l = p$; also $c = -2\,p$, $d = +2\,p$, as found by placing $x = p$ in the equation of the parabola and solving for y. Observing finally that $x = \dfrac{y^2}{4\,p}$ in the present instance, we have as the volume desired

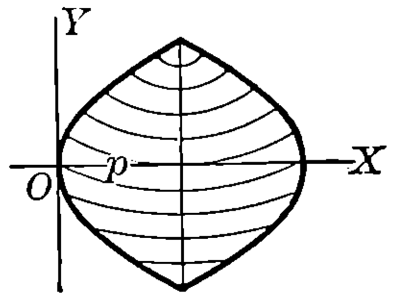

FIG. 136

$$V = \pi\int_{-2p}^{2p}\left(\frac{y^2}{4\,p} - p\right)^2 dy = \pi\int_{-2p}^{2p}\left(\frac{y^4}{16\,p^2} - \frac{y^2}{2} + p^2\right) dy$$

$$= \pi\left[\frac{y^5}{80\,p^2} - \frac{y^3}{6} + p^2y\right]_{y=-2p}^{y=2p} = \frac{32}{15}\pi p^3.$$

EXERCISES

1. Find by integration the volume of the right cone generated by revolving about the X-axis the line-segment joining the origin to the point (p, q).
Ans. $\frac{1}{3}\pi pq^2$.

2. Find the volume of the sphere generated by revolving the circle $x^2 + y^2 = r^2$ about a diameter. *Ans.* $\frac{4}{3}\pi r^3$.

3. Find the volume of the segment of a sphere of radius r cut out by two parallel planes whose distances from the center are respectively a and b.
Ans. $\frac{1}{3}\pi(b - a)(3r^2 - a^2 - b^2 - ab)$.

4. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis. *Ans.* $\frac{4}{3}\pi ab^2$.

5. Find the volume generated by revolving one arch of the sinusoid, $y = \sin x$, about the X-axis. *Ans.* $\frac{1}{2}\pi^2$.

6. The area under the curve $y = \sec x$ between $x = 0$ and $x = \frac{1}{4}\pi$ is revolved about the X-axis. Find the volume generated. *Ans.* π.

7. Find the volume generated by revolving about the X-axis the area in the second quadrant under the curve $y = e^x$. *Ans.* $\frac{1}{2}\pi$.
HINT. See Art. 136.

8. Find the volume generated by revolving about the X-axis the area which is bounded by the hyperbola $xy = a^2$ and the X-axis and which lies to the right of the line $x = a$. *Ans.* πa^3.

9. Find the volume generated by revolving about the X-axis the area inclosed by the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{32}{105}\pi a^3$.

10. Find the volume generated by revolving about the X-axis the area inclosed between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. *Ans.* $\frac{1}{15}\pi a^3$.

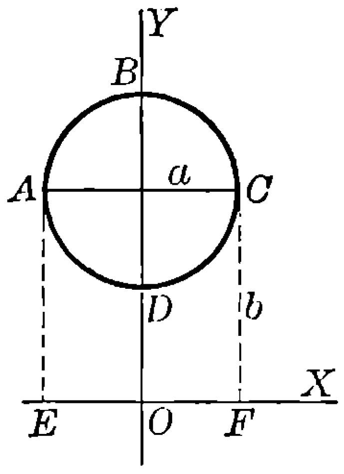

FIG. 137

11. Find the volume of the ring (torus) generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about the X-axis.
Ans. $2\pi^2 a^2 b$.

HINT. Note that the volume desired will be that generated by the area $EABCF$ (Fig. 137) minus that generated by the area $EADCF$.

12. Find the volume generated by revolving one arch of the sinusoid, $y = \sin x$, about the Y-axis. *Ans.* $2\pi^2$.

13. The area between the cissoid $y^2 = \frac{x^3}{2a - x}$ and its asymptote $x = 2a$ is revolved about the asymptote. Find the volume generated.
Ans. $2\pi^2 a^3$.

14. Find the volume generated by revolving about the Y-axis the area between the curve $y = e^{-\frac{1}{2}x^2}$ and its asymptote. *Ans.* 2π.

15. Find the volume generated by revolving an ellipse of semiaxes a, b about the tangent at the end of the minor axis. *Ans.* $2\pi^2 ab^2$.

16. A round hole of radius a is bored through the center of a sphere of radius $2a$. Find the volume cut out. *Ans.* $\frac{4}{3}\pi(8 - 3\sqrt{3})a^3$.

17. A segment of height h is cut from the side of a sphere of radius r. Find its volume. *Ans.* $\frac{1}{3}\pi h^2(3r - h)$.

18. One arch of the cycloid

$$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta), \end{cases}$$

is revolved about its base. Find the volume generated. *Ans.* $5\pi^2a^3$.

19. The hypocycloid

$$\begin{cases} x = a\cos^3\theta, \\ y = a\sin^3\theta, \end{cases}$$

is revolved about the X-axis. Find the volume generated. *Ans.* $\frac{32}{105}\pi a^3$.

142. A theorem on limits. In order to show the further applications of integration to geometry and physics, it is desirable at this point to note the following general theorem in the theory of limits. The proof, which belongs to a more advanced course, we shall omit.

Theorem. *Let*

$$(\alpha) \qquad \alpha_1(n),\ \alpha_2(n),\ \alpha_3(n),\ \cdots,\ \alpha_i(n), \qquad \alpha_n(n)$$

be a sequence of n expressions each dependent upon n and positive for every value of n. Suppose also that each expression approaches the limit zero (is an infinitesimal) as $n \to \infty$

Consider now a second sequence of n expressions

$$(\beta) \qquad \beta_1(n),\ \beta_2(n),\ \beta_3(n),\ \cdots,\ \beta_i(n), \qquad \beta_n(n)$$

bearing the same description and such that, whatever the value of i, we have

$$\lim_{n\to\infty}\frac{\alpha_i(n)}{\beta_i(n)} = 1.$$

Then we may write

$$\lim_{n\to\infty}\left\{\alpha_1(n) + \alpha_2(n) + \quad + \alpha_n(n)\right\} =$$

$$\lim_{n\to\infty}\left\{\beta_1(n) + \beta_2(n) + \quad + \beta_n(n)\right\},$$

*provided these two limiting sums exist.**

* Strictly speaking, this theorem (known as *Duhamel's theorem*) when stated with full mathematical accuracy places further conditions than we have here specified upon the expressions (infinitesimals) $\alpha_i(n)$, $\beta_i(n)$. Such further conditions are, however, chiefly of theoretic interest inasmuch as they are usually satisfied *ipso facto* in the usual applications when the simple conditions already described are satisfied. Under these circumstances, it seems best to state the theorem as above, relegating the more precise statement to a more advanced and theoretical treatment of the calculus.

For example, consider the two sequences

$$(\alpha) \qquad \frac{1}{n^2}, \frac{2}{n^2}, \frac{3}{n^2}, \quad \frac{n}{n^2},$$

$$(\beta) \qquad \frac{n+1}{n^3}, \frac{2n+1}{n^3}, \frac{3n+1}{n^3}, \quad \frac{n^2+1}{n^3}.$$

Each expression in the (α) sequence and each in the (β) sequence evidently is positive for every value of n and each approaches zero as $n \to \infty$ Moreover, in the (α) sequence the ith term is $\alpha_i = \frac{i}{n^2}$ while in the (β) sequence it is $\beta_i = \frac{in+1}{n^3}$, so that

$$\lim_{n\to\infty} \frac{\alpha_i}{\beta_i} = \lim_{n\to\infty} \frac{\frac{i}{n^2}}{\frac{in+1}{n^3}} = \lim_{n\to\infty} \frac{in}{in+1} = 1.$$

The theorem therefore applies and it asserts that

$$\lim_{n\to\infty} \left\{ \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \quad + \frac{n}{n^2} \right\}$$
$$= \lim_{n\to\infty} \left\{ \frac{n+1}{n^3} + \frac{2n+1}{n^3} + \frac{3n+1}{n^3} + \quad + \frac{n^2+1}{n^3} \right\}.$$

Note. In the simple example just cited, each of the two limits appearing may be actually evaluated by elementary algebra. Thus, the first limit may be written in the form

$$\lim_{n\to\infty} \frac{1}{n^2} \left\{ 1 + 2 + 3 + \quad + n \right\} = \lim_{n\to\infty} \frac{1}{n^2} \left\{ \frac{n}{2}(1+n) \right\} = \lim_{n\to\infty} \frac{1+n}{2n} = \frac{1}{2}, \qquad (1)$$

and similarly the second limit may be put in the form

$$\lim_{n\to\infty} \left\{ \left(\frac{1}{n^2} + \frac{1}{n^3}\right) + \left(\frac{2}{n^2} + \frac{1}{n^3}\right) + \left(\frac{3}{n^2} + \frac{1}{n^3}\right) + \quad \left(\frac{n}{n^2} + \frac{1}{n^3}\right) \right\}$$
$$= \lim_{n\to\infty} \left\{ \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \quad + \frac{n}{n^2} \right\} + \lim_{n\to\infty} \left\{ \frac{n}{n^3} \right\}$$
$$= \frac{1}{2} + \lim_{n\to\infty} \frac{1}{n^2} = \frac{1}{2} + 0 = \frac{1}{2}. \qquad (2)$$

From (1) and (2) it appears that the two limits are equal, as the theorem asserts.

The student should observe that in this as in all other applications of the theorem the number of terms in both the (α) and (β) sequences increases indefinitely as $n \to \infty$, each individual term in each sequence meanwhile approaching zero.

143. Areas of surfaces of revolution. The area generated by revolving about the X-axis an arc of a given curve $y = f(x)$ may now be formulated as follows:

Theorem. *The surface S generated by revolving about the X-axis the arc of the curve $y = f(x)$ between the abscissas $x = a$ and $x = b$ is given by the formula*

$$S = 2\pi\int_a^b y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx.$$

PROOF. Divide the interval (a, b) into n equal parts each of length Δx and draw the ordinates to the curve at the various points of division. Let P and Q be any two adjacent points as thus determined on the curve and draw the line-segment PQ joining them. Then, as the curve revolves about the X-axis, the line PQ generates the surface of the frustum of a cone of revolution and the sum of all the areas thus arising from all such lines PQ between $x = a$ and $x = b$ forms a good approximation when n is large to the area generated by the curvilinear arc itself, which is the area in question.

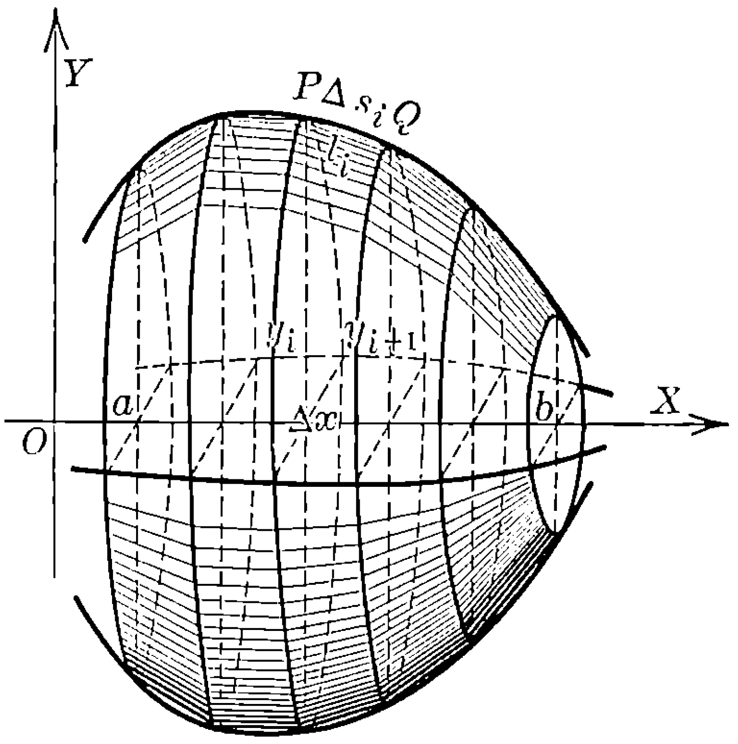

FIG. 138

This premised, let y_0 be the ordinate to the curve at $x = a$ and let $y_1, y_2, \cdots, y_{n-1}$ be the ordinates at the various points of division of the interval (a, b). Then the area, A_i, of the frustum generated by the revolution of the ith line-segment is equal, by solid geometry, to the length of this line, which we will call l_i, multiplied by half the sum of the circumferences of the two circular bases; that is, we have

$$A_i = l_i \times \tfrac{1}{2}(2\pi y_i + 2\pi y_{i+1}) = \pi(y_i + y_{i+1})l_i.$$

This expression approaches zero as $n \to \infty$ inasmuch as $\lim_{n\to\infty} l_i = 0$, but the limit of the *sum* of such expressions, which is the thing desired, is not readily formulated by direct use of the fundamental theorem of Art. 138. For this reason, we now proceed as follows, arriving at the desired goal by use of the theorem on limits stated in the last article.

Let Δs_i be the arc corresponding to the chord l_i and consider the new expression

$$B_i = 2\pi y_i \Delta s_i.$$

For this we have $\lim_{n\to\infty} B_i = 0$ and we have also

$$\lim_{n\to\infty} \frac{A_i}{B_i} = \lim_{n\to\infty} \frac{\pi(y_i + y_{i+1})}{2\pi y_i} \frac{l_i}{\Delta s_i} = \lim_{n\to\infty} \frac{y_i + y_{i+1}}{2 y_i} \lim_{n\to\infty} \frac{l_i}{\Delta s_i}.$$

Of the two limits here appearing on the extreme right, the first is equal to 1 inasmuch as y_i and y_{i+1} approach coincidence as $n\to\infty$, and the second limit likewise is 1 inasmuch as it is the limiting ratio of a chord to its corresponding arc. Hence, the extreme left member is equal to 1. In accordance with the theorem of the last article, we may therefore substitute for the limit of the *sum* of the expressions A_i the limit of the *sum* of the expressions B_i.

Finally, let us consider still a third expression, namely,

$$C_i = 2\pi y_i \sqrt{1 + \left(\frac{dy_i}{dx}\right)^2} \Delta x,$$

where $\frac{dy_i}{dx}$ indicates the value of $\frac{dy}{dx}$ when $y = y_i$. By Art. 49 we obtain

$$\lim_{n\to\infty} \frac{B_i}{C_i} = \lim_{n\to\infty} \frac{\Delta s_i}{\sqrt{1 + \left(\frac{dy_i}{dx}\right)^2} \Delta x} = 1.$$

Hence, again applying the theorem of the last article, we may say that the limit of the sum of the C_i is the same as that of the B_i and hence the same as that of the A_i. But the limit of the sum of the C_i is, by the fundamental theorem of Art. 138, equal to

$$2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

The proof is thus complete.

Similarly, if an arc of the curve $y = f(x)$ is revolved about the Y-axis, the surface generated is given by the formula

$$S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

where (c, d) is the interval along the Y-axis under the arc. The proof of this formula, being similar to the foregoing, is left to the student as an exercise.

Example. The portion of the parabola $y^2 = 4\,px$ lying between the origin and the line $x = p$ is revolved about the X-axis. Find the area of the surface generated. (See Fig. 135.)

SOLUTION. Here we have $\frac{dy}{dx} = \frac{2\,p}{y}$ so that the theorem gives

$$S = 2\,\pi \int_0^p y \sqrt{1 + \frac{4\,p^2}{y^2}}\,dx = 2\,\pi \int_0^p \sqrt{y^2 + 4\,p^2}\,dx$$

$$= 2\,\pi \int_0^p \sqrt{4\,px + 4\,p^2}\,dx = \frac{4\,\pi}{p}\,\frac{2}{3}\Big[(px + p^2)^{\frac{3}{2}}\Big]_{x=0}^{x=p} = \frac{8}{3}\,\pi p^2(2\sqrt{2} - 1).$$

EXERCISES

1. Find by integration the area of the surface of the sphere generated by revolving the circle $x^2 + y^2 = r^2$ about a diameter. *Ans.* $4\,\pi r^2$.

2. Show that the surface of the zone of a sphere lying between two parallel planes at distance h apart is the same wherever the zone be selected, and is equal to $2\,\pi\, rh$.

3. Find by integration the area of the surface of the cone generated by revolving about the X-axis the line-segment joining the origin to the point (a, b).
Ans. $\pi b\sqrt{a^2 + b^2}$.

4. Work Ex. 3 in case the line is revolved about the Y-axis.
Ans. $\pi a\sqrt{a^2 + b^2}$.

5. Show by integration that the lateral area of the right circular cylinder of radius r and altitude h is $2\,\pi\, rh$.

6. Find the surface generated by revolving about the X-axis an arch of the sinusoid $y = \sin x$. See Fig. page 368. *Ans.* $2\,\pi[\sqrt{2} + \log(\sqrt{2} + 1)]$.

7. Find the area of the surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis. *Ans.* $\frac{12}{5}\,\pi a^2$.

8. Find the surface of the ring (torus) generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about the X-axis. Compare Ex. 11, page 270. *Ans.* $4\,\pi^2\,ab$.

9. Find the surface generated by revolving an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base. Compare Ex. 18, page 271.
Ans. $\frac{64}{3}\,\pi a^2$.

10. The portion of the curve $y = e^{-x}$ lying in the first quadrant is revolved about the X-axis. Find the area generated. *Ans.* $\pi[\sqrt{2} + \log(1 + \sqrt{2})]$.

11. The arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from $x = 0$ to $x = a$ is revolved about the X-axis. Find the area generated. *Ans.* $\frac{1}{4}\,\pi a^2(e^2 + 4 - e^{-2})$.

12. Work Ex. 11 when the revolution is about the Y-axis.
Ans. $2\,\pi a^2(1 - e^{-1})$.

13. The ellipse $3\,x^2 + 4\,y^2 = 3$ is revolved about its major axis. Find the area of the surface (prolate spheroid) generated. *Ans.* $\pi\left(\frac{3}{2} + \frac{\pi}{\sqrt{3}}\right)$.

14. Work Ex. 13 when the revolution is about the minor axis, thus giving an oblate spheroid. *Ans.* $\frac{\pi}{2}(4 + 3 \log 3)$.

15. Find the area of the surface generated by revolving each of the following arcs about the X-axis.

(*a*) $y = x^3$ from $x = 0$ to $x = 1$;

(*b*) $y^2 = 9x$ from $x = 0$ to $x = 3$;

the loop of the curve $8a^2y^2 = x^2(a^2 - x^2)$.

16. A closed curve which has an axis of symmetry is revolved about a line parallel to this axis but not intersecting the curve. Prove that the area of the ring generated is equal to the area of the right cylinder whose base is the area inclosed by the curve and whose altitude is the length of the circumference described by any point in the axis of symmetry.

144. Further geometric applications. Besides the foregoing applications of the definite integral to the determination of arc-lengths, volumes, and areas, there is a wide variety of other possible geometric applications. For these no general rule or theorem can be given, but the nature of such applications is readily understood from an illustrative example.

Example 1. An equilateral triangle ABC of varying size moves across a circle of radius r in such a way that (*a*) the plane of the triangle remains perpendicular to a certain diameter DE of the circle and (*b*) the two vertices A, B remain upon the circumference. Find the volume of the solid generated.

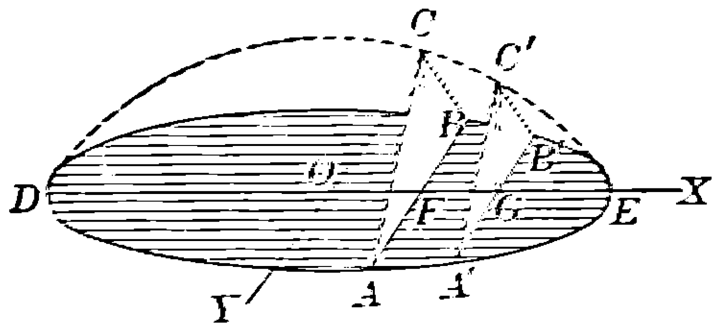

FIG. 139

SOLUTION. Imagine the solid to be cut by a large number of parallel planes which are at a common distance d apart and which are all perpendicular to DE. The solid will thus be divided into a large number of slices such as $ABCA'B'C'$ each having parallel bases such as ABC and $A'B'C'$, which are equilateral triangles. Now, having selected any one of these slices, let us imagine that upon its base ABC a right prism is erected having FG as altitude. Although such a prism may evidently protrude outside the slice $ABCA'B'C'$ still the volumes of the prism and the slice will be nearly equal when FG is small. Likewise, the *sum* of all such prisms corresponding to the n cross sections ABC will form, when n is large, a good approximation to the actual volume of the solid in question. Finally, the *limit* of this sum of prisms as $n \to \infty$, and hence as $FG = d \to 0$, may be regarded as this actual volume itself. This being clear, it remains but to formulate this sum analytically and to ascertain, if possible, its limit as $n \to \infty$.

For this purpose, take DE as the X-axis and take as Y-axis the line OY drawn in the plane of the circle through the center O perpendicular to DE. Then the equation of the circle becomes

$$x^2 + y^2 = r^2. \tag{1}$$

Now let each of the equal parts into which DE is divided by the n cross sections be represented by Δx and suppose that F is, in particular, the point on DE corresponding to the ith cross section. Then ABC becomes in particular the position of the varying triangle when $x = x_i = OF$. Finally, call V_i the volume of the prism of altitude $FG = \Delta x$ erected upon ABC. Then, by solid geometry we have

$$V_i = \textit{area}\ ABC \times \Delta x = \tfrac{1}{2} AB \times FC \times \Delta x = AF \times FC \times \Delta x. \tag{2}$$

Moreover, ABC being equilateral, *angle* $CAB = 60°$, so that

$$FC = AC \quad \sin CAB = AC \quad \sin 60° = \frac{\sqrt{3}}{2} AC$$

$$= \frac{\sqrt{3}}{2} AB = \frac{\sqrt{3}}{2} 2\,AF = \sqrt{3}\,AF.$$

Substituting this value of FC in (2), we have

$$V_i = \sqrt{3} \quad \overline{AF}^2 \quad \Delta x. \tag{3}$$

Now, AF being the ordinate of the point A whose abscissa is $x = OF = x_i$, we find from (1) that

$$\overline{AF}^2 = r^2 - x_i^2.$$

Thus, (3) takes the form

$$V = \sqrt{3}(r^2 - x_i^2)\,\Delta x.$$

Finally, the desired limit as $n \to \infty$ of the *sum* of all such expressions V_i as i ranges from 1 to n, thus causing x to range from $-r$ to $+r$, is, by the theorem of Art. 138,

$$V = \sqrt{3} \int_{-r}^{r} (r^2 - x^2)\,dx = \sqrt{3}\left[r^2 x - \frac{x^3}{3}\right]_{x=-r}^{x=+r} = \frac{4\sqrt{3}}{3} r^3. \tag{4}$$

This, therefore, is the volume sought.

As a generalization of the method employed in the foregoing solution, let us now suppose that a solid S of any shape is given and that it is required to determine its volume V Suppose that an X-axis can be so chosen that the area A of the various cross sections perpendicular to this axis becomes a known, or determinate, function of x,

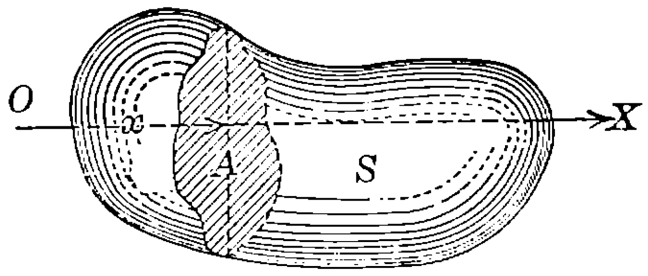

FIG. 140

where x is the abscissa of the point of intersection of the cross section with the same axis; that is, suppose that

$$A = f(x).$$

Then the method may be applied and the required volume V is found to be expressed by the formula (compare (4))

$$V = \int_a^b f(x)\,dx, \tag{5}$$

where the limits of integration, a, b, are respectively the least and greatest values of x pertaining to S. The student is advised to establish this general result for himself by following through the various steps of reasoning analogous to those of the foregoing special example.

Thus, if in Fig. 139 the moving triangle ABC, instead of being always equilateral, is always an isosceles right triangle with right angle at C, then, using the same axes as before, the area A of ABC becomes expressed as a function of $x = OF$ as follows:

$$\begin{aligned} A &= \tfrac{1}{2} AB \times FC = AF \times FC = AF \times AF \tan \angle FAC \\ &= \overline{AF}^2 \tan 45^\circ = (r^2 - \overline{OF}^2) \times 1 = r^2 - x^2 = f(x). \end{aligned}$$

Hence, by (5), the volume generated in this case is

$$V = \int_{-r}^{r} (r^2 - x^2)\,dx = \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4}{3} r^3.$$

The following example will illustrate how the foregoing methods may be used also in other ways:

Example 2. Having given that the circumference of a circle of radius a is $2\pi a$, show by integration that the area is πa^2.

SOLUTION. Regard the circular area as made up of a series of concentric rings each of width Δr. Then the area of the ring whose inner radius is r is equal approximately * to $2\pi r\,\Delta r$. Moreover, if all the products of this character arising from the various rings be added, the limit of this sum as the number of rings is increased indefinitely will be the area sought, and this limit, by the fundamental theorem of Art. 138, is recognized to be

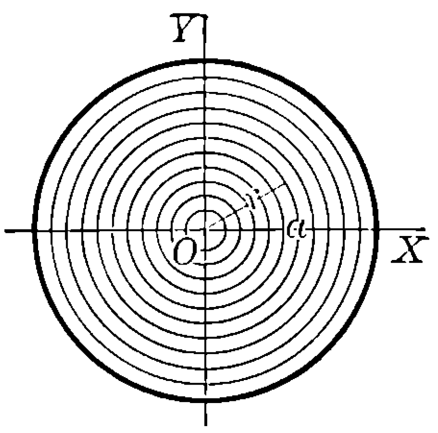

FIG. 141

$$A = \int_0^a 2\pi r\,dr = 2\pi \int_0^a r\,dr = 2\pi \left[\frac{r^2}{2} \right]_0^a = \pi a^2.$$

* Two infinitesimals will hereafter be termed "approximately" equal when, as each approaches zero, the limit of their ratio is 1.

EXERCISES

1. Work illustrative Example 1, Art. 144, in case the moving triangle is an isosceles right triangle with right angle at A, other conditions remaining as before. *Ans.* $\frac{8}{3}r^3$.

2. Work illustrative Example 1, Art. 144, in case the base of the solid, instead of being the circle $x^2 + y^2 = r^2$, is the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, other conditions remaining as before. *Ans.* $\frac{4}{3}\sqrt{3}\,ab^2$

3. Work illustrative Example 1, Art. 144, in case the moving triangle, instead of being equilateral, is always an isosceles triangle of constant altitude h, other conditions remaining as before. The solid generated in this case is called a *conoid.* See Fig. 142. *Ans.* $\frac{1}{2}\pi r^2h$.

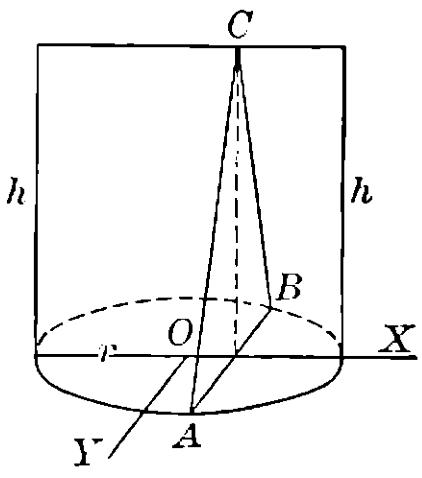

FIG. 142

4. Work Ex. 3 in case a circle passing through the three points A, B, C (Fig. 142) is used instead of the triangle ABC. *Ans.* $\pi r\left(\frac{h^2}{2} + \frac{2\,r^2}{3} + \frac{4\,r^4}{15\,h^2}\right)$.

5. Upon the area bounded by the parabola $y^2 = 4\,px$ and the line $x = p$, a solid is erected whose cross-sections perpendicular to the X-axis are all equilateral triangles. Find the volume of such a solid. *Ans.* $2\sqrt{3}\,p^3$.

HINT. The area of the equilateral triangle of side a is $\frac{\sqrt{3}}{4}a^2$.

6. Work Ex. 5 in case the cross-sections are all isosceles right triangles with the right angle directly above the X-axis, other conditions remaining as before. *Ans.* $2\,p^3$.

7. Having given that the area of a sphere of radius a is $4\,\pi a^2$, show by integration that the volume of the sphere is $\frac{4}{3}\pi a^3$. Compare illustrative Example 2, Art. 144.

8. By regarding a sphere as the limit of the sum of a series of parallel circular plates, show that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

9. It being given that the area of the ellipse whose semiaxes are a, b is πab, prove that the volume of the ellipsoid whose semiaxes are a, b, c is $\frac{4}{3}\pi abc$. Compare Ex. 8.

10. Upon that portion of the parabola $y^2 = 4\,px$ which lies in the first quadrant circles are described having their centers upon the curve and their planes perpendicular to the axis of the parabola, the radius of any one circle being half the ordinate of its center. Find the volume of the curved trumpet-like solid of which all such circles form parallel cross-sections, it being assumed that the solid extends from $x = 0$ to $x = h$. *Ans.* $\frac{1}{2}\pi ph^2$.

11. Work Ex. 10 in case the curve used is the portion of the circle $x^2 + y^2 = r^2$ lying in the first quadrant, other conditions remaining as before except that $h = r$. *Ans.* $\frac{1}{6}\pi r^3$.

12. A circle of varying size is moved in such a way that its center remains upon a straight line while its plane remains perpendicular to that line. At the

end of any time t, measured in minutes, the distance of the center from its starting point is $2t$ inches while the value of the radius at this instant is $3t^2$ inches. Find the volume of the solid generated by the circle during the first T minutes of its motion. *Ans.* $\frac{18}{5}\pi T^5$ cu. in.

13. Show that the result in Ex. 12 is the same in case the center of the moving circle, instead of being confined to the straight line, is confined to any continuous curve passing through the origin, other conditions remaining the same.

14. A rectangle moves from a fixed point, one side being always equal to the distance of the plane of the rectangle from this point and the other side being equal to the square of this distance. What is the volume generated while the rectangle moves a distance of 3 feet?

145. Applications to physics. Besides the applications of the definite integral in geometry, as illustrated in the foregoing articles, there are numerous applications in physics. We shall confine ourselves to three of these, as indicated below:

(*a*) ***Fluid pressure.*** In Fig. 143 let $NOLM$ be a vertical retaining wall (such as the gate in a water-channel) against which a fluid stands at rest and let it be required to determine the total horizontal pressure exerted by the fluid against the wall. The problem may evidently be approached as follows:

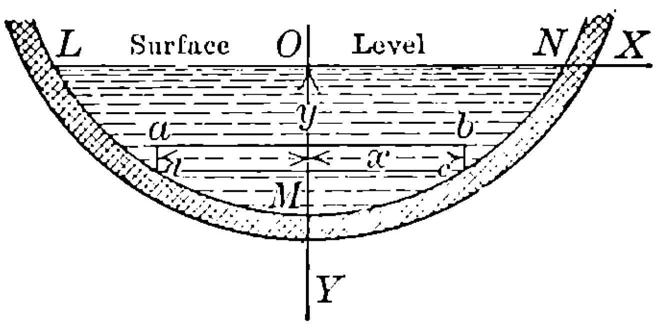

FIG. 143

Divide the extreme depth OM into a large number of equal parts and corresponding to each part draw a rectangle, as $abcd$, inscribed within the contour of the wall and having its long edges parallel to the surface of the fluid. Calculate the pressure upon each such rectangle and take the sum of the results. Finally, determine the limit of this sum as the number of divisions of OM, and hence the number of inscribed rectangles, increases indefinitely. This limit will be the required total pressure on the wall. This method being clear, it remains to express its steps analytically, thus arriving at an actual formula.

Take O as origin of coördinates and let the X-axis be taken along the horizontal surface of the fluid. For convenience, take the (positive) Y-axis vertically downward and at first let us confine the attention to the pressure against the portion OMN of the wall; that is, the portion within which x is positive. Let Δy be the width of each of the rectangles $abcd$ and suppose that the weight of a unit volume of the fluid is w. Then, by elementary physics, the pressure

upon the portion of *abcd* in question will be equal to w times its area multiplied by the depth, y, of its mid-section (see dotted line); that is, will be equal to $w \cdot x \quad \Delta y \quad y = wxy\Delta y$, where x is the length of the portion in question.* The limit of the sum of such products pertaining to the various rectangles will therefore be the desired pressure P upon the entire area lying to the right of OY But, by the familiar argument based upon Arts. 138 and 142, this limit is given by the formula

$$P = w\int_0^h xy\,dy, \tag{1}$$

where h = the extreme depth OM. Here x is, of course, a function of y which is determined by a knowledge of the equation of the bounding curve LMN of the reservoir. The equation of this curve must, therefore, be given in the form $x = f(y)$, or be converted to this form before the integration is performed.

Similarly, the pressure against the portion of the wall to the *left* of OY may be calculated by integration. However, in case OY divides the area $NOLM$ symmetrically, as in Fig. 143 and as often happens in practice, it suffices to calculate the pressure over the right portion NOM and then double the result.

Example 1. A trough whose cross-section is an equilateral triangle with sides equally inclined to the vertical is filled with water to a depth of 3 feet. Given that a cubic foot of water weighs approximately 62 lbs., find the pressure on one end of the trough.

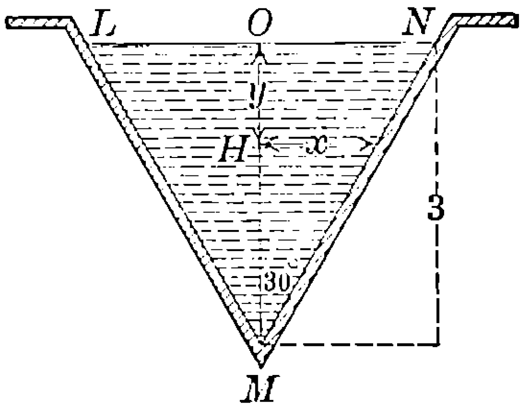

FIG. 144

SOLUTION. Here

$$x = MH \quad \tan\angle OMN = (3 - y)\tan 30° = \frac{3 - y}{\sqrt{3}}.$$

Hence, by use of (1) we obtain as the pressure upon NOM

$$P = 62\int_0^3 \frac{(3 - y)y}{\sqrt{3}}dy = \frac{62}{\sqrt{3}}\left[\frac{3\,y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{279}{\sqrt{3}} = 161.085 \text{ lbs.}$$

Allowing for the equal pressure upon the portion LOM, the total pressure required is therefore

$$2 \times 161 = 322 \text{ lbs. approximately.}$$

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* In general, the pressure upon *any* area will be equal to w times this area multiplied by the *average* depth of all its points beneath the surface.

(*b*) ***Amount of illumination.*** The following example illustrates the application of integration to this subject.

Example 2. In Fig. 145 a circular floor of radius a is being illuminated by a source of light L situated on the vertical axis of the circle at a distance h above the center. Given that the intensity of illumination varies inversely as the square of the distance from the source, find how far the light should be lowered in order that the total illumination on the floor may be doubled.

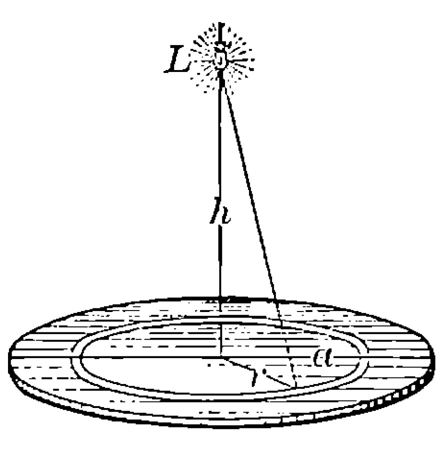


FIG. 145

SOLUTION. Consider a circular ring of inner radius r and of narrow width Δr lying on the surface of the floor and concentric with it. The distance of all points in this ring from L is approximately $\sqrt{h^2 + r^2}$ and hence, in accordance with the law above stated, the intensity of the illumination at all points in this ring is approximately

$$\frac{k}{h^2 + r^2},$$

where k is a constant whose value is the intensity of the illumination at a unit's distance from L. Observing now that the area of the ring is approximately $2\,\pi r \times \Delta r$, it follows that the amount of illumination on the ring is approximately

$$\frac{k}{h^2 + r^2} \times 2\,\pi r \times \Delta r = \frac{2\,\pi k r\,\Delta r}{h^2 + r^2}.$$

The sum of such expressions as r assumes the series of values pertaining to a set of rings such as just described covering the entire floor will therefore form a good approximation to the total amount of illumination I received. Moreover, the limit of this sum as the number of rings is indefinitely increased, and hence as $\Delta r \to 0$, will be I itself. But, by the usual argument, this limit is expressible as a definite integral, thus giving

$$I = 2\,\pi k \int_0^a \frac{r\,dr}{h^2 + r^2} = \pi k \Big[\log (h^2 + r^2)\Big]_{r=0}^{r=a}$$

$$= \pi k [\log (h^2 + a^2) - \log h^2] = \pi k \log \frac{h^2 + a^2}{h^2}. \qquad (2)$$

This general result having been obtained, we may now consider the special problem in hand. Let x be the required distance through which L should be lowered. Then, the illumination at the new level will be obtained from (2) by replacing h by $h - x$. If this be done and the result placed equal to twice the original value, as the problem requires, we obtain the following equation for determining x:

$$\pi k \log \frac{(h-x)^2+a^2}{(h-x)^2} = 2\,\pi k \log \frac{h^2+a^2}{h^2}.$$

Dividing both members by πk and recalling that twice the logarithm of any quantity is equal to the logarithm of the square of the quantity, this equation becomes

$$\log \frac{(h-x)^2+a^2}{(h-x)^2} = \log \left(\frac{h^2+a^2}{h^2}\right)^2\cdot$$

Hence, we must have

$$\frac{(h-x)^2+a^2}{(h-x)^2} = \left(\frac{h^2+a^2}{h^2}\right)^2$$

or

$$1 + \frac{a^2}{(h-x)^2} = \left(1+\frac{a^2}{h^2}\right)^2,$$

from which we obtain

$$(h-x)^2 = \frac{h^4}{2\,h^2+a^2}.$$

This equation is readily solved for x and gives as the required answer

$$x = h - \frac{h^2}{\sqrt{2\,h^2+a^2}}.$$

(*c*) ***Work.*** If a weight of w pounds be lifted through a vertical distance of d feet, the work done is defined in physics to be *wd foot-pounds.* More generally, if a point P be moved a distance of d feet in direct opposition to the action of a constant force of F pounds, the work done is Fd foot-pounds. In case the force F is *not* constant but changes from point to point during the motion, the work done becomes expressed, in conformity with the foregoing definition, as a definite integral in the manner illustrated below.

Example 3. An elastic cord, such as a long rubber band, is stretched out 2 feet beyond its natural (unstretched) length and the force which it then exerts at one end is found to be 8 ounces. Find how much work has been done during the stretching, it being given by physics (*Hooke's law*) that the tension in a stretched elastic cord varies directly as the amount of the extension beyond the natural length.

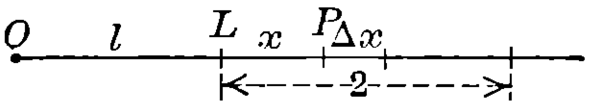


Fig. 146

Solution. Let $l = OL$ be the natural length of the cord and suppose one end to be held fast at O while the other end P is carried a distance x beyond L. Then, by the foregoing law, the force F at P will be given by the equation

$$F = kx, \tag{3}$$

where k is some constant. But it is given that $F = 8$ oz. $= \frac{1}{2}$ lb. when

$x = 2$ ft. Placing these special values of F and x in (3), we find $k = \frac{1}{4}$, so that (3) takes the specific form

$$F = \tfrac{1}{4}x. \tag{4}$$

It follows that if the cord be extended a small distance Δx beyond P, the work done will be approximately $\frac{1}{4}x \cdot \Delta x$ and hence, applying the argument already familiar from former examples, the total work done in extending the cord 2 feet beyond its natural length will be

$$\frac{1}{4}\int_0^2 x\,dx = \frac{x^2}{8}\Big]_0^2 = \frac{1}{2}\text{ foot-pound.}$$

EXERCISES

1. Work illustrative Example 1, Art. 145, in case the cross-section of the trough is a right triangle instead of an equilateral triangle, other conditions remaining as before. *Ans.* 558 lbs. approximately.

2. Work illustrative Example 1, Art. 145, in case the trough is semicircular in cross-section with radius of 3 feet, other conditions remaining as before. *Ans.* 1116 lbs.

3. In illustrative Example 1, Art. 145, show that if the water-level be raised 1 foot the total end-pressure will be increased by more than 137 per cent.

4. A narrow paved alley 400 feet long separating two city stores is illuminated by an arc light placed midway from the ends and 20 feet from the ground. Show that the total illumination received by the pavement is given approximately by the formula

$$I = \frac{kd}{10}\arctan 10,$$

where k is the intensity of the light at a foot's distance from it, and where d is the width of the alley.

5. The interior of an art-gallery is circular in form, the radius of the circle being a. If the height of the room is h, show that the amount of illumination received by the side-wall from a center light in the ceiling is given by the formula

$$I = 2\pi k \arctan\frac{h}{a},$$

where k is the intensity of the light at a unit's distance from it.

6. By means of formula (2) obtained for illustrative Example 2, page 282, and of the formula of Ex. 5 show that if the floor and the side wall of a circular room are to receive equal amounts of illumination from a center light in the ceiling, the room should be so constructed that the ratio of the height to the radius of the base, that is, the ratio $R = \frac{h}{a}$, shall satisfy the equation

$$2\arctan R = \log\left(1 + \frac{1}{R^2}\right).$$

The root of this equation, as determined approximately by trial, is $R = 0.67$.

7. Newton's law of universal gravitation states that the gravitational attraction between two material particles varies directly as the product of their masses and inversely as the square of the distance between them. Hence, letting A represent the attraction between two particles whose masses are m_1 and m_2 situated d units apart, we have

$$A = k\frac{m_1 m_2}{d^2},$$

where k is a constant whose value is the attraction between two unit particles situated a unit's distance apart. Employing this law, show that the attraction of a slender straight wire of length l upon a unit particle situated in the direction of the wire at a distance d from the nearest end is given by the formula

$$A = \frac{k\rho l}{d(d+l)},$$

where ρ is the density (mass per unit length) of the wire.

8. Using the law stated in Ex. show that the gravitational attraction of a straight wire of length l upon a unit particle situated upon the perpendicular bisector of the wire at a distance h from the midpoint of the wire is

$$A = k\rho h\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx}{(x^2+h^2)^{\frac{3}{2}}} = \frac{2\,k\rho l}{h(l^2+4\,h^2)^{\frac{1}{2}}},$$

where k and ρ have the meanings indicated in Ex. 7.

9. If the density ρ of a sphere varies directly as the distance r from the center, that is, $\rho = kr$, where k is a constant, show that the mass of the sphere of radius a is given by the formula

$$M = \pi k a^4.$$

10. Assuming that the force of attraction of a magnetic pole upon a small steel ball varies inversely as the square of the distance of the ball from the pole and that at the distance d this force is f, show that the work done in removing the ball from the distance d to the distance D is given by the formula

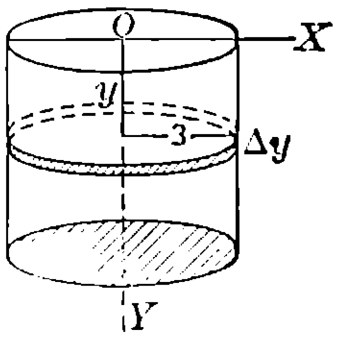
$$W = fd^2\left[\frac{1}{d} - \frac{1}{D}\right].$$

11. Assuming that the length of the trough described in illustrative Example 1, Art. 145, is 4 feet, determine the work required to pump all the water to the original surface level. *Ans.* 1288 ft. lbs.

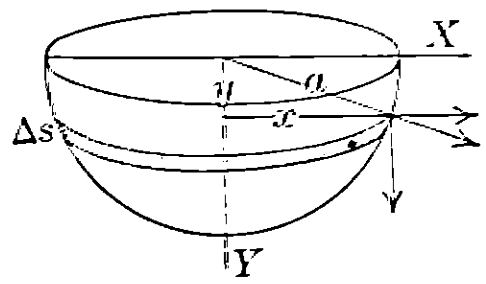
Hint. First formulate the work required, approximately, to pump up the strip of water of width Δy lying at a depth of y.

12. Find the work required to pump to the surface the water in the trough described in Ex. 1, it being given that the trough is 4 feet long. *Ans.* 2232 ft. lbs.

13. A cylindrical pail 6 inches in diameter and 1 foot high is full of water. Calculate the total pressure on the convex surface of the pail. *Ans.* $\frac{31}{2}\pi$ lbs.



Hint. Consider the surface of the pail as made up of a large number of narrow bands each of width Δy and of radius 3 inches. Consider the limit of the sum of the pressures on all these bands as their number is indefinitely increased.



14. A hemispherical bowl of radius a feet is full of water. Calculate (a) the total horizontal pressure on the bowl; (b) the total vertical pressure; (c) the total pressure normal to the surface.

Ans. Horizontal pressure = Vertical pressure = $\frac{124}{3}\pi a^3$ lbs. Normal pressure = $62\pi a^3$ lbs.

HINT. Consider the surface of the bowl as made up of a large number of narrow circular bands each of width Δs, then proceed analogously to Ex. 13. Recall the formula $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ of Art. 49.

15. By use of the result for Ex. 8 find a formula for the gravitational attraction of a square of side l and density ρ upon a unit particle situated in the plane of the square at a distance d from a side and upon the perpendicular bisector of that side.

16. A cylindrical pail of radius a and height h is filled with water. How long will it take the water to leak out through a hole of area S in the bottom of the pail, it being given that the velocity of discharge of a fluid through a horizontal orifice is that acquired by a body falling freely from a height equal to the depth of the water.

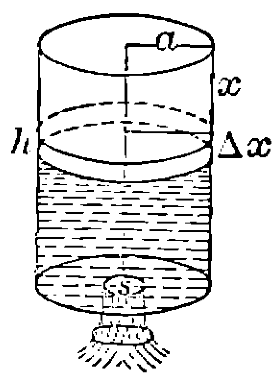


FIG. 147

OUTLINE OF SOLUTION. Using the notation of Fig. 147 and recalling that the velocity acquired by a freely falling body in descending any distance s is $\sqrt{2\,gs}$, it follows from the data of the problem that at the instant the water level reaches x the velocity of discharge through the hole is $\sqrt{2\,g(h-x)}$. Hence, in a small interval of time Δt immediately following this instant, the amount of water discharged is approximately $S\sqrt{2\,g(h-x)}\,\Delta t$. This amount must be equal to the loss from the surface above, which is seen to be $\pi a^2\,\Delta x$, where Δx is the descent of the water level during Δt. We therefore have approximately

$$S\sqrt{2\,g(h-x)}\,\Delta t = \pi a^2\,\Delta x,$$

$$\Delta t = \frac{\pi a^2}{S\sqrt{2\,g(h-x)}}\,\Delta x.$$

It follows by the usual reasoning based on Arts. 138 and 142 that the required time is

$$T = \frac{\pi a^2}{S\sqrt{2\,g}}\int_0^h \frac{dx}{\sqrt{h-x}} = \frac{2\,\pi a^2}{S}\sqrt{\frac{h}{2\,g}}.$$

17. If in Ex. 16 the water is contained in an inverted cone of height h, radius of base a, and with the opening S at the vertex, show that the time of discharge is

$$T = \frac{2\,\pi a^2}{5\,S}\sqrt{\frac{h}{2\,g}}.$$

18. Work Ex. 16 in case the water is contained in a hemisphere of radius a, the opening S being at the lowest point.

CHAPTER XVI

SUCCESSIVE INTEGRATION

146. Definition. Let $f(x, y)$ be a function of the two independent variables x, y. Suppose for the moment that x is held fixed in value while the other variable, y, is allowed to change. Then $f(x, y)$ becomes a function of y alone. As such, let us suppose that it is now integrated between the limits $y = c$ and $y = d$ that is, let us consider the integral

$$\int_c^d f(x, y)\, dy.$$

The value of this integral will naturally depend upon the value originally assigned to x, hence it may be regarded as a function of x. Considered as such, let us now integrate it between the limits $x = a$ and $x = b$; that is, let us consider the expression

$$\int_a^b \left[\int_c^d f(x, y)\, dy\right] dx.$$

This expression, which is usually written in the more compact form

$$\int_a^b \int_c^d f(x, y)\, dy\, dx,$$

is called the **double iterated integral,** or briefly the **double integral,** of $f(x, y)$ from $y = c$ to $y = d$ and from $x = a$ to $x = b$.

Example. Evaluate the double integral

$$\int_0^3 \int_1^2 (x^2 + y^2)\, dy\, dx.$$

SOLUTION. Performing the first, or y integration, remembering that in this we are to regard x as a constant, we obtain

$$\int_1^2 (x^2 + y^2)\, dy = \left[x^2 y + \tfrac{1}{3} y^3\right]_{y=1}^{y=2} = 2\,x^2 + \tfrac{8}{3} - x^2 - \tfrac{1}{3} = x^2 + \tfrac{7}{3}.$$

Performing now upon this result the second, or x integration, we obtain as the final desired result

$$\int_0^3 (x^2 + \tfrac{7}{3})\, dx = \left[\tfrac{1}{3} x^3 + \tfrac{7}{3} x\right]_{x=0}^{x=3} = 9 + 7 - 0 = 16.$$

147. Generalization. In the definition of the last article it was tacitly assumed that the limits of integration a, b, c, d were all fixed numbers. It is conceivable, however, and it often happens in the applications which are to follow, that the limits c, d of y pertaining to the *first* integration may depend upon the variable x which plays a part in the *second* integration. That the double integral continues to have a definite meaning in such cases is illustrated in the following example (compare with example in Art. 146):

Example. Evaluate the double integral

$$\int_0^3\int_x^{2x}(x^2+y^2)\,dy\,dx.$$

SOLUTION. Performing the first or y-integration gives

$$\int_x^{2x}(x^2+y^2)\,dy=[x^2y+\tfrac{1}{3}y^3]_{y=x}^{y=2x}=2\,x^3+\tfrac{8}{3}\,x^3-x^3-\tfrac{1}{3}\,x^3=\tfrac{10}{3}\,x^3.$$

Performing upon this result the second or x-integration gives as the final result

$$\frac{10}{3}\int_0^3 x^3\,dx=\frac{10}{3}\left[\frac{x^4}{4}\right]_{x=0}^{x=3}=\frac{135}{2}.$$

Still more generally, the notion of a double integral is immediately extended to that of a **triple integral.** Such an integral, in fact, is one of the form

$$\int_a^b\int_c^d\int_e^f F(x,\,y,\,z)\,dz\,dy\,dx.$$

Here $F(x, y, z)$ is a function of the three independent variables x, y, z. As to the various limits of integration, e and f, or the limits of the *first* or z-integration, may depend upon either x or y or both inasmuch as these letters appear in the subsequent integrations; similarly, c and d, or the limits of the *second* or y-integration, may depend upon x, while a and b, or the limits of the *third* or x-integration, are necessarily fixed numbers; that is, are independent of either x, y, or z.

Example. Evaluate the triple integral

$$\int_0^1\int_{-x}^{x}\int_{x-y}^{x+y}(x+y+z)\,dz\,dy\,dx. \qquad (1)$$

SOLUTION. Performing the first or z-integration, recalling that in this we are to consider x and y as constants, we obtain

$$\int_{x-y}^{x+y}(x+y+z)\,dz=\left[xz+yz+\frac{z^2}{2}\right]_{z=x-y}^{z=x+y}=x(x+y)+y(x+y)$$
$$+\frac{(x+y)^2}{2}-x(x-y)-y(x-y)-\frac{(x-y)^2}{2}=4\,xy+2\,y^2.$$

Performing upon this result the second or y-integration, recalling that in this we are to consider x as a constant, we obtain

$$\int_{-x}^{x}(4\,xy + 2\,y^2)\,dy = \left[2\,xy^2 + \frac{2\,y^3}{3}\right]_{y=-x}^{y=x} = 2\,x^3 + \frac{2}{3}\,x^3 - 2\,x^3 + \frac{2}{3}x^3 = \frac{4}{3}x^3.$$

Finally, performing upon this the third or x-integration, we obtain as the desired value of (1)

$$\int_0^1 \frac{4}{3}\,x^3\,dx = \frac{4}{3}\left[\frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1}{3}.$$

Quadruple integrals, quintuple integrals, etc. are now readily defined by analogy with the foregoing. We shall, however, not be concerned with these in the present book.

EXERCISES

Evaluate each of the following iterated integrals:

1. $\int_0^2\int_{-1}^1 xy\,dy\,dx.$ *Ans.* 0. **2.** $\int_{-1}^1\int_0^2 (x^2 - y^2)\,dy\,dx.$ *Ans.* -4.

3. $\int_2^3\int_0^x (x^2 - y^2)\,dy\,dx.$ *Ans.* $\frac{65}{6}$. **4.** $\int_0^{\frac{\pi}{2}}\int_0^x y\sin x\,dy\,dx.$ *Ans.* $\frac{\pi}{2} - 1.$

5. $\int_0^3\int_0^2\int_0^1 x\,yz\,dz\,dy\,dx.$ *Ans.* **6.** $\int_0^1\int_0^x\int_0^{x+y} x\,yz\,dz\,dy\,dx.$ *Ans.* $\frac{17}{144}$.

7. $\int_2^3\int_{-1}^1\int_0^1 (xy + xz + yz)\,dz\,dy\,dx.$ *Ans.* $\frac{5}{2}$.

8. $\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_0^{mx} dz\,dy\,dx.$ *Ans.* $\frac{1}{3}\,r^3m.$

9. $\int_0^\pi\int_0^{a(1+\cos\theta)} \rho^2\sin\theta\,d\rho\,d\theta.$ *Ans.* $\frac{4}{3}\,a^3.$

10. $\int_0^\pi\int_0^{a\cos\theta} \rho\sin\theta\,d\rho\,d\theta.$ *Ans.* $\frac{1}{3}\,a^2.$

11. $\int_{-a}^a\int_0^{\frac{y^2}{a}} (x + y)\,dx\,dy.$ *Ans.* $\frac{1}{5}\,a^3.$ **12.** $\int_0^a\int_0^x \frac{y\,dy\,dx}{x^2 + y^2}.$ *Ans.* $\frac{a}{2}\log 2.$

13. $\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2\sqrt{x^2 + y^2}\,dy\,dx.$ *Ans.* $\frac{1}{20}\,\pi a^5.$

14. $\int_0^a\int_0^x \frac{x^2\,dy\,dx}{\sqrt{x^2 + y^2}}.$ *Ans.* $\frac{a^3}{3}\log(1 + \sqrt{2}).$

15. Evaluate each of the following integrals:

(*a*) $\int_0^1\int_{-1}^1 xy^2\,dy\,dx;$ (*d*) $\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^a \rho^4\,d\rho\,d\theta;$

(*b*) $\int_0^1\int_{-x}^x xy^2\,dy\,dx;$ (*e*) $\int_0^1\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx;$

(*c*) $\int_0^1\int_0^{\frac{\pi}{2}} \rho^2\sin\theta\,d\theta\,d\rho;$ (*f*) $\int_1^2\int_0^z\int_0^{x\sqrt{3}} \frac{x\,dy\,dx\,dz}{x^2 + y^2}.$

I. Applications to Geometry

148. Area between two curves. Rectangular coördinates. Let two curves $y = f(x)$ and $y = \phi(x)$ be given and let it be required to find the area inclosed between them and the ordinates drawn across the curves at $x = a$ and $x = b$. This area, as we shall now show, may be formulated as a double iterated integral in the manner indicated in the following theorem:

Theorem. *The area A bounded by the curves $y = f(x)$, $y = \phi(x)$ and the lines $x = a$, $x = b$ is given by the formula*

$$A = \int_a^b \int_{f(x)}^{\phi(x)} dy\, dx. \tag{1}$$

Proof. Between $x = a$ and $x = b$ draw a set of straight lines perpendicular to the X-axis and at an equal small distance Δx apart. Draw similarly a set of lines perpendicular to the Y-axis and at an equal small distance Δy apart. The area A in question is thus divided into a large number of interior rectangles, each of area $\Delta y\, \Delta x$, together with a number of portions of such rectangles lying along the boundary. Consider now the strip made up of these rectangles and parts of rectangles lying between any two adjacent perpendiculars to the X-axis such as the perpendiculars $x = x$ and $x = x + \Delta x$. The side pq of this strip (see Fig. 148) we may evidently regard as the limit as $\Delta y \to 0$ of the *sum* of all the lengths Δy lying between the levels at which $y = f(x)$ and $y = \phi(x)$ respectively, x being understood to keep the fixed value $x = x$ under consideration. By the theorem of Art. 138 we may therefore write

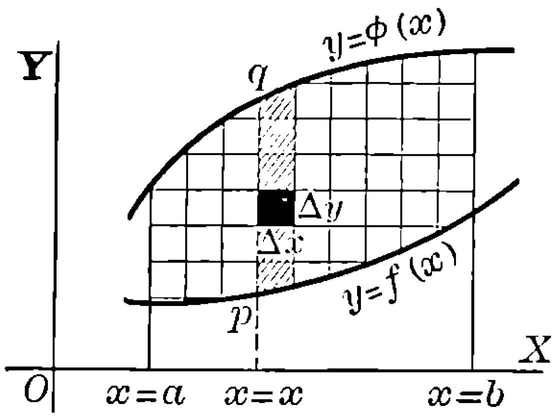


Fig. 148

$$pq = \int_{f(x)}^{\phi(x)} dy. \tag{2}$$

As to the *area* of the strip, it is equal approximately* to $pq\ \Delta x$ and hence by (2) is equal approximately to

$$\left[\int_{f(x)}^{\phi(x)} dy\right] \Delta x. \tag{3}$$

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* See footnote, page 278.

It follows by an application of the theorem of Art. 142 that the *sum* of the areas of all the strips, which is the required area A, may be regarded as the limit of the sum of all the approximations (3) as the number of strips is indefinitely increased, Δx meanwhile approaching the limit zero and x ranging from its least value $x = a$ to its greatest value $x = b$. But this latter limit, by applying the theorem of Art. 138 to (3), takes the form

$$\int_a^b \left[\int_{f(x)}^{\phi(x)} dy\right] dx, \quad \text{or} \quad \int_a^b \int_{f(x)}^{\phi(x)} dy\, dx.$$

The proof is thus complete.

Corollary. In the foregoing discussion the equations of the two given curves were regarded as of the forms $y = f(x)$, $y = \phi(x)$ respectively. If, on the other hand, these equations are present in the forms $x = F(y)$, $x = \Phi(y)$, then the formula for the area A bounded by the two curves and the two lines $y = c$, $y = d$ may be conveniently expressed in the form

$$A = \int_c^d \int_{F(y)}^{\phi(y)} dx\, dy. \tag{4}$$

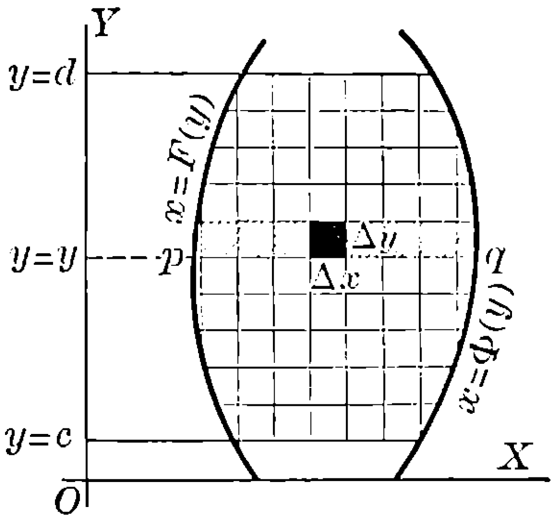

FIG. 149

For the proof, we need only to divide A as before into rectangles of area $\Delta x\, \Delta y$, but now consider a typical strip parallel to the X-axis instead of the Y-axis; then consider the limit of the sum of such strips. The details, being similar to those in the former case, are left to the student.

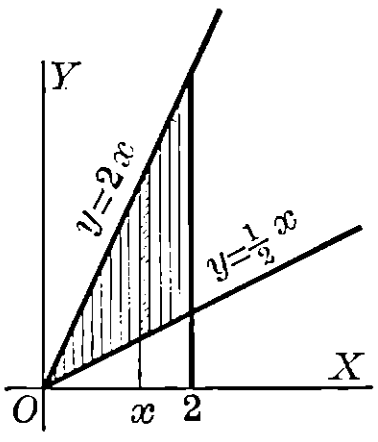

FIG. 150

Example. Find by double integration the area of the triangle included between the lines $y = \frac{1}{2}x$, $y = 2x$, and $x = 2$.

SOLUTION. Applying formula (1), noting that within the area A in question y ranges from the value $x/2$ to $2x$ as one passes along the perpendicular to the X-axis erected at any typical point $x = x$ on that axis, we have

$$A = \int_0^2 \int_{\frac{x}{2}}^{2x} dy\, dx = \int_0^2 \left[2x - \frac{x}{2}\right] dx = \frac{3}{2}\int_0^2 x\, dx = \frac{3}{2}\left[\frac{x^2}{2}\right]_{x=0}^{x=2} = 3.$$

The same example, if done by use of formula (4) instead of (1), would be worked as follows: First write the equations of the two lines in the

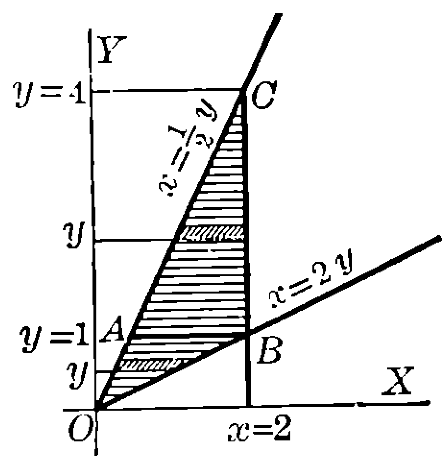

FIG. 151

forms $x = 2y$ and $x = \frac{1}{2}y$. Then regard the entire area of OBC as made up of the parts OBA and ABC. Now, the areas of OBA and ABC are each readily formulated by (4). Doing this and adding the results gives

$$A = \int_0^1 \int_{\frac{1}{2}y}^{2y} dx\,dy + \int_1^4 \int_{\frac{1}{2}y}^{2} dx\,dy$$

$$= \tfrac{3}{2}\int_0^1 y\,dy + \int_1^4 (2 - \tfrac{1}{2}y)\,dy = \tfrac{3}{4} + 2\tfrac{1}{4} = 3.$$

149. Area between two curves. Polar coördinates. We may now formulate the area bounded by two curves whose equations are given in polar coördinates as follows:

Theorem. *The area A bounded by the curves $\rho = f(\theta)$, $\rho = \phi(\theta)$ and the radii vectores $\theta = \alpha$, $\theta = \beta$ is given by the formula*

$$A = \int_\alpha^\beta \int_{f(\theta)}^{\phi(\theta)} \rho\,d\rho\,d\theta. \qquad (1)$$

PROOF. Between the fixed radii vectores $\theta = \alpha$ and $\theta = \beta$ draw a set of radii vectores at equal small angular distances $\Delta\theta$ apart; also from the pole O as center draw a set of concentric circles at equal small radial distances $\Delta\rho$ apart. The area A in question is thus divided into a large number of interior figures $PQRS$, each approximately rectangular in shape, together with a number of portions of such figures lying along the boundary. Here $PQ = \Delta\rho$ and, calling ρ the radius vector of P, $PS = \rho\ \Delta\theta$. Hence, approximately, we have

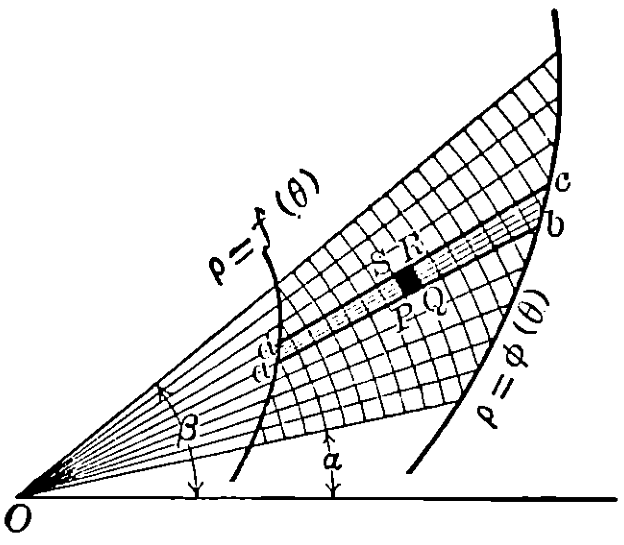

FIG. 152

$$\text{area } PQRS = PQ \times PS = \rho\,\Delta\rho\,\Delta\theta. \qquad (2)$$

It follows that approximately

$$\text{area } abcd = \Sigma\rho\,\Delta\rho\,\Delta\theta = \Delta\theta\,\Sigma\rho\,\Delta\rho, \qquad (3)$$

where the symbol Σ (read sigma) indicates that the *sum* of all the products (2) corresponding to the various points P along ab is being taken. But as $\Delta\rho \rightarrow 0$ the expression (3) approaches, by Art. 138, as its limit the value

$$\Delta\theta \int_{f(\theta)}^{\phi(\theta)} \rho\,d\rho. \qquad (4)$$

From (3) and (4) we readily infer (though a rigorous proof lies beyond the present text) that approximately (in the sense indicated in the footnote of page 278) we may write

$$\textit{area abcd} = \Delta\theta \int_{f(\theta)}^{\phi(\theta)} \rho\, d\rho. \tag{5}$$

The required area A, which is the sum of all such areas $abcd$, may therefore, by Art. 142, be regarded as the limit as $\Delta\theta \to 0$ of the sum of the expressions (5) as θ ranges from its initial value $\theta = \alpha$ to its final value $\theta = \beta$. But the limit of this sum, by Art. 138, is

$$\int_\alpha^\beta \left[\int_{f(\theta)}^{\phi(\theta)} \rho\, d\rho\right] d\theta, \quad \text{or} \quad \int_\alpha^\beta \int_{f(\theta)}^{\phi(\theta)} \rho\, d\rho\, d\theta.$$

The proof is thus complete.

Corollary. In the foregoing discussion the equations of the two given curves were regarded as of the forms $\rho = f(\theta)$, $\rho = \phi(\theta)$. If, on the other hand, these equations are present in the forms $\theta = F(\rho)$, $\theta = \Phi(\rho)$, then the formula for the area A bounded by the two curves and the two arcs $\rho = r$, $\rho = s$ may be conveniently expressed in the form

$$A = \int_r^s \int_{F(\rho)}^{\Phi(\rho)} \rho\, d\theta\, d\rho. \tag{6}$$

The proof, being analogous to the foregoing, is readily suggested by reference to Fig. 153. It is therefore left as an exercise for the student.

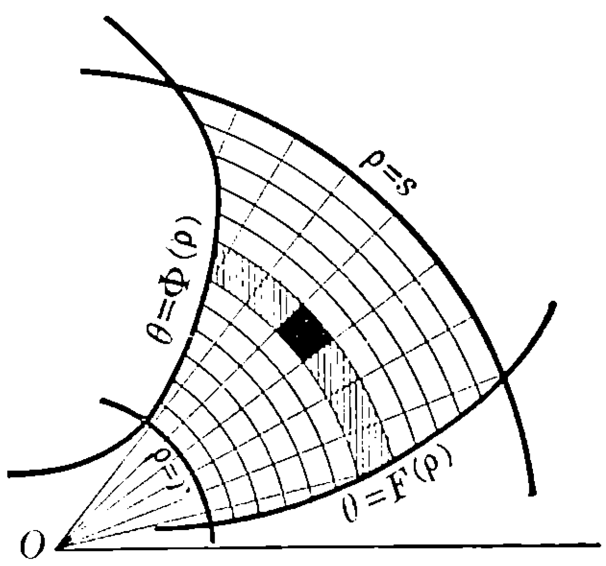

FIG. 153

Example. Find by double integration the area of the crescent included between the circles $\rho = a \cos \theta$ and $\rho = 2\, a \cos \theta$.

SOLUTION. Applying formula (1), noting that θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ within the crescent, we have at once for the desired area

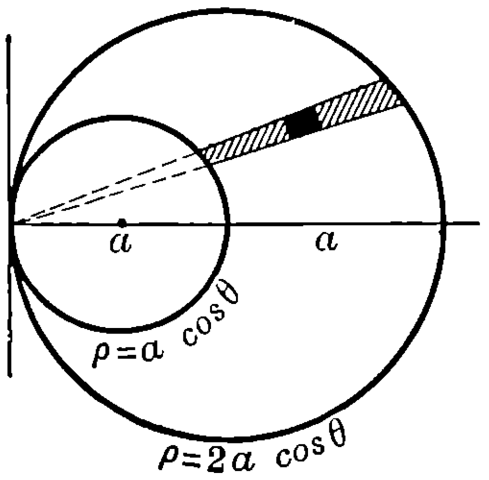

FIG. 154

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{a\cos\theta}^{2a\cos\theta} \rho\, d\rho\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\rho^2}{2}\right]_{\rho = a\cos\theta}^{\rho = 2a\cos\theta} d\theta = \frac{3\, a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{3\, a^2}{2}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\,\theta\right]_{\theta = -\frac{\pi}{2}}^{\theta = \frac{\pi}{2}} = \frac{3}{4}\pi a^2.$$

EXERCISES

1. Find by double integration the area included between the parabola $y^2 = 4\,px$ and the line $y = x$, integrating first with respect to y. Draw a figure.

Ans. $\int_0^{4p}\int_x^{\sqrt{4px}} dy\,dx = \frac{8}{3}p^2.$

2. Work Ex. 1, integrating first with respect to x.

3. Find by double integration the area included between the semicubical parabola $y^2 = x^3$ and the line $y = x$. Draw a figure and work by integrating with respect to y first. *Ans.* $\frac{1}{10}$.

4. Work Ex. 3, integrating with respect to x first.

5. Find by double integration the area between the two circles $\rho = a\cos\theta$, $\rho = b\cos\theta$ $(b > a)$. Draw a figure and integrate with respect to ρ first.

Ans. $\frac{\pi}{4}(b^2 - a^2)$.

6. Find by double integration the area included between the circle $\rho = 2\,a\cos\theta$ and the cardioid $\rho = a(1 + \cos\theta)$. *Ans.* $\frac{1}{2}\pi a^2$.

7. Find by double integration the areas bounded by each of the following pairs of curves:

(a) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}},\ x + y = a;$ *Ans.* $\frac{1}{3}a^2$.

(b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},\ x + y = a;$ *Ans.* $\frac{1}{2}a^2 - \frac{3}{32}\pi a^2$.

(c) $x^2 + y^2 = 2,\ y^2 = x^3;$ *Ans.* $\frac{\pi}{2} - \frac{1}{5}$.

(d) $y^2 = ax,\ y^2 = \dfrac{x^3}{2\,a - x};$ *Ans.* $\frac{8}{3}a^2 - \frac{3}{4}\pi a^2$.

(e) $x^2 - y^2 = a^2,\ x^2 + y^2 = b^2\ (b > a)$.

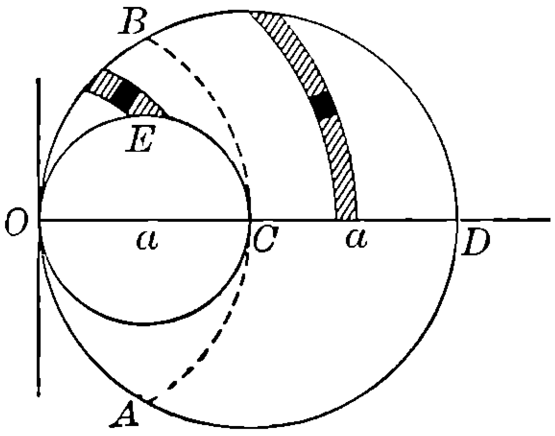

FIG. 155

8. Work the illustrative example of Art. 149 when the first integration performed is with respect to θ.

HINT. Draw the arc ACB (Fig. 155) and consider separately the areas BCD and $OBCE$. Add the two results and double the answer in order to include the portion of the required area *below* OD.

9. Work Ex. 5 when the first integration performed is with respect to θ.

10. Find by double integration the area of the crescent formed between the parabola $\rho = a\sec^2\frac{\theta}{2}$ and the circle $\rho = 2\,a$. *Ans.* $2\,\pi a^2 - \frac{8}{3}a^2$.

11. The curve $y = x^4 - 4\,x^3$ has two points of inflection. Find the area bounded by the curve and the inflectional tangent which passes through the origin. Find also the area bounded by the curve and the other inflectional tangent. *Ans.* 51.2, 51.2.

150. Volume under a surface. Suppose that a surface is given whose equation is of the form $z = f(x, y)$; suppose also that in the XY-plane beneath this surface there is given a certain area A bounded by a closed curve T Upon A as a base and with T as directrix let a right cylinder be erected having all its elements parallel to the Z-axis and suppose that the portion of the given surface $z = f(x, y)$ which this cylinder intercepts is indicated by S. Then the volume of this cylinder included between A and S (otherwise described as the volume under the surface $z = f(x, y)$ above A) may be determined through the following theorem:

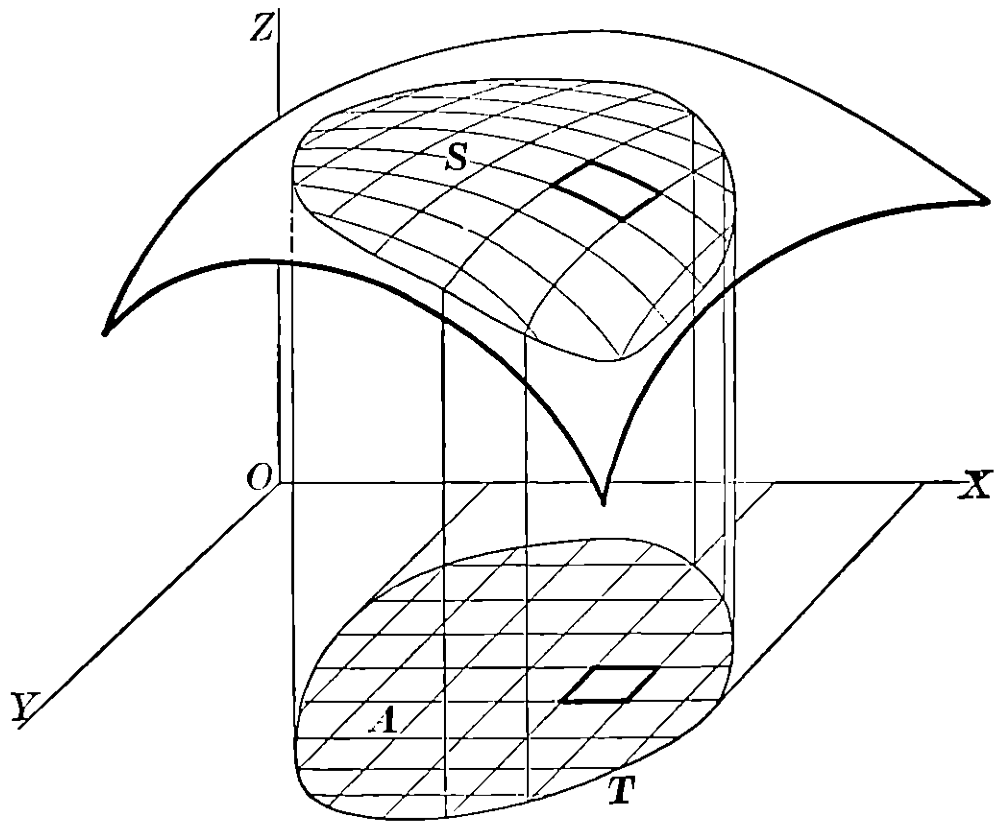

FIG. 156

Theorem. *The volume under the surface $z = f(x, y)$ above the area A in the XY-plane is determined by the formula*

$$V = \int\int f(x, y)\, dy\, dx,$$

where the limits of the first, or y-integration are respectively the least and the greatest value of y lying upon the line within A parallel to the Y-axis at a distance x from this axis, while the limits of the second, or x-integration are respectively the least and the greatest value of x within A.

The student will observe upon a careful reading of this theorem that in order to apply it in any given case one must know not only the equation $y = f(x, y)$ of the surface of which S is a part, but he must also know the equation of the plane curve T in the XY-plane, the latter being required in order to determine the limits of integration to be used. See illustration, page 298.

PROOF. Within A draw a set of lines parallel to the Y-axis at a distance Δx apart and similarly draw a set of lines parallel to the X-axis at a distance Δy apart (compare Art. 148). Select any one of the rectangles, as PQ, thus formed in A and let P denote that one of its vertices which lies nearest the origin O. Upon this rectangle as a base erect a rectangular parallelepiped $PQRU$ whose altitude PR shall be the value of $f(x, y)$ when x and y are the coördinates of P This parallelepiped will have at least one point R lying upon the

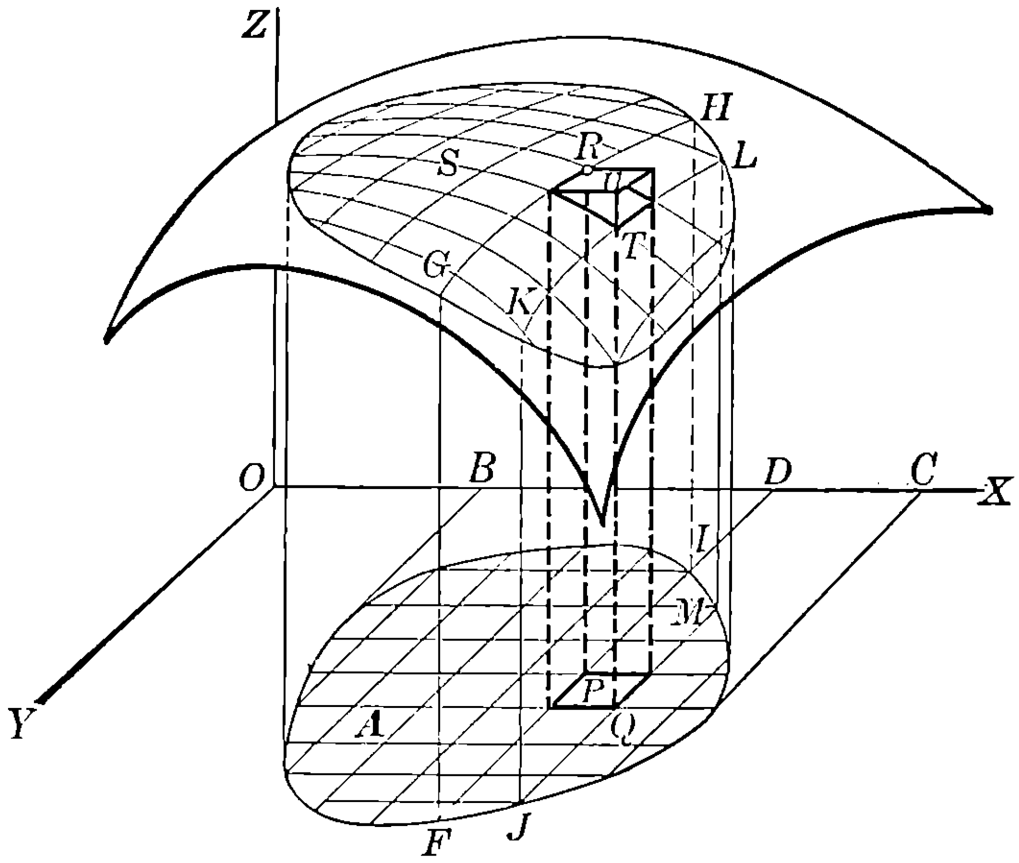

FIG. 157

given surface and its volume, when Δx and Δy are small, will evidently form a good approximation to the volume under the given surface above the rectangle PQ. Suppose now that this process is similarly carried out for *every* rectangle PQ within A and that the sum of the volumes of all the resulting parallelepipeds is taken. Evidently this sum will form a good approximation to the total volume above A in question, and the limit of this sum as $\Delta x \to 0$ and $\Delta y \to 0$ may be regarded as the actual value of the desired volume. It remains, then, but to formulate this entire process analytically.

The parallelepiped $PQRU$ has a base PQ whose area is $\Delta y \cdot \Delta x$ and an altitude PR equal to $f(x, y)$. Hence,

$$\textit{volume of parallelepiped } PQRU = f(x, y)\, \Delta y\, \Delta x. \qquad (1)$$

Consider now all the parallelepipeds $PQRU$ standing upon the strip $FIMJ$ of A. For each of these the value of x to be used in (1) is the same, namely, $x = OD$, while the value of y differs, in general, from one parallelepiped to the next. The sum of the volumes of

all these parallelepipeds has a limit as $\Delta y \to 0$ which, by the familiar reasoning of foregoing articles, is equal to the following expression (compare (3), Art. 148)

$$\left[\int_{y_1}^{y_2} f(x, y)\,dy\right] \Delta x, \tag{2}$$

where the limits of integration, y_1 and y_2, represent respectively the least and greatest value of y along the line FI, these limits depending in general upon the value of $x = OD$ that has been chosen. This being premised, it remains but to formulate the sum of all the expressions (2) as x is allowed to take on its various values belonging to the edges of the different strips $FIMJ$ within A, and finally to formulate the limit of this sum as $\Delta x \to 0$. The final result is thus seen to be

$$\int_a^b \left[\int_{y_1}^{y_2} f(x, y)\,dy\right] dx, \quad \text{or} \quad \int_a^b \int_{y_1}^{y_2} f(x, y)\,dy\,dx,$$

where $a = OB$ and $b = OC$ are respectively the least and greatest values of x within A.

This result, being coextensive with that stated in the theorem, completes the proof.*

Corollary. Consideration of the foregoing reasoning shows at once that if, in the formula for V as given in the theorem, the order of integration be reversed, that is, if the first integration be taken with respect to x and the second with respect to y, the theorem continues to hold provided that, as the limits of the x-integration we take respectively the least and the greatest value of x lying upon the line within A parallel to the X-axis at a distance y from this axis, while the limits of the y-integration are respectively the least and the greatest value of y within A.

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* Strictly speaking, it might be questioned at this point whether the limit of the sum of the expressions (1) when *first* $\Delta y \to 0$ and *secondly* $\Delta x \to 0$ (as was done in the foregoing proof) gives the same result as if the processes were followed in the opposite order; namely, first $\Delta x \to 0$ and then $\Delta y \to 0$. In fact, how do we know that the same limiting sum results no matter how Δx and Δy are mutually related when both approach zero? The answer can only be given after a more advanced course in calculus. Suffice it to say that, under conditions upon $f(x, y)$ which are not seriously limitative, it can be shown that the limit in question has a value independent of the manner in which both $\Delta x \to 0$ and $\Delta y \to 0$.

A similar theoretical question is involved in the proofs of other theorems subsequently occurring in this chapter.

Example. Find the volume of the solid lying in the first octant and bounded by the plane $z = x + y$, the cylinder $x^2 + y^2 = 1$, and the XY-plane.

SOLUTION. The volume in question lies above a quadrant of the circle $x^2 + y^2 = 1$ in the XY-plane within which the y-limits upon a line distant x from the Y-axis are 0 and $\sqrt{1 - x^2}$, while the x-limits over the entire quadrant are 0 and 1. Hence, applying the theorem, we have for the required volume

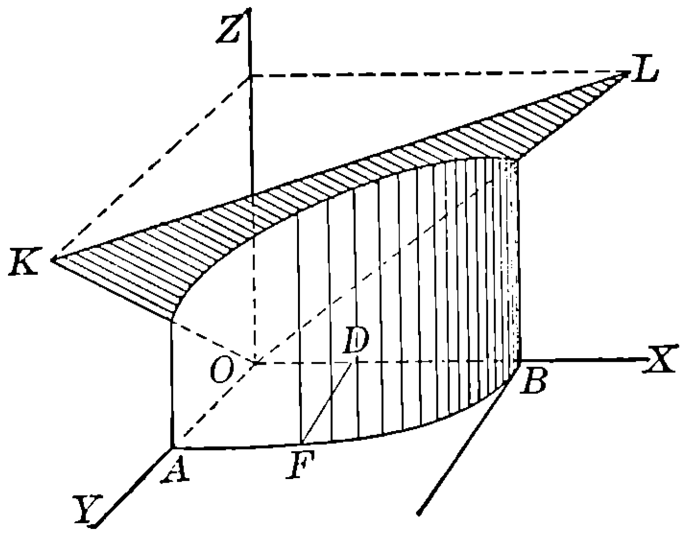

FIG. 158

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} (x + y)\, dy\, dx$$
$$= \int_0^1 x\sqrt{1 - x^2}\, dx + \tfrac{1}{2}\int_0^1 (1 - x^2)\, dx$$
$$= \left[-\frac{1}{3}(1 - x^2)^{\frac{3}{2}} + \frac{x}{2} - \frac{x^3}{6} \right]_0^1 = \frac{2}{3}.$$

EXERCISES

1. Work the illustrative Example of Art. 150 in case the arc AB (Fig. 158), instead of being circular, is an arc of the parabola $y^2 = 4(1 - x)$. *Ans.* $1\frac{8}{15}$.

2. Work the last example when the first integration is taken with respect to x and the second with respect to y. See Corollary, Art. 150.

3. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by Art. 150.

4. Find the volume of the tetrahedron bounded by the coördinate planes and the plane $x/a + y/b + z/c = 1$. *Ans.* $\frac{1}{6}\, abc$.

5. Find the volume common to the two cylinders $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$. *Ans.* $\frac{16}{3} r^3$.

6. Find the volume of the solid contained between the paraboloid of revolution $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2\,ax$, and the XY-plane. *Ans.* $\frac{3}{2}\pi a^3$.

7. Find the volume of the cylindrical column standing upon the area common to the two parabolas $y = x^2$, $x = y^2$ as a base and intercepted by the surface $z = 4 + y - x^2$. *Ans.* $\frac{587}{420}$.

8. Prove that the entire volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is $\dfrac{4}{3}\pi abc$.

9. Find the volume bounded by the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{4}{35}\pi a^3$.

10. Formulate each of the following volumes as double (iterated) integrals without, however, attempting to evaluate them:

(*a*) the volume cut from a sphere of radius a by a right circular cylinder of radius b ($b < a$) whose axis passes through the center of the sphere;

(*b*) the volume cut from a sphere of radius a by a right circular cylinder of radius b ($b < a$) whose axis is tangent to the surface of the sphere;

(*c*) the volume of the right cylindrical solid standing upon the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ as a base and intercepted by the cone $z^2 = x^2 + y^2$.

151. Definition. Whenever the limits of integration pertaining to a double integral are supplied either in the manner described in the theorem of Art. 150, or in the Corollary of that article, we shall say hereafter that they have been supplied so as to **cover** the entire area A.

152. Area of a surface. Returning to the general problem discussed in Art. 150, let us now suppose that, the conditions remaining as before, it is required to find the area S intercepted from the given surface $z = f(x, y)$ by the right cylinder erected upon A as a base (see Fig. 156). In this connection we shall establish the following theorem:

Theorem. *The area S intercepted from the surface $z = f(x, y)$ by the right cylinder erected upon the closed area A lying in the XY-plane is determined by the formula*

$$S = \int\int\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dy\, dx,$$

where the limits for the double integral are to be supplied so that the integration shall cover (see Art. 151) *the entire area A.*

PROOF. Proceed as in the proof of the theorem of Art. 150 to draw two sets of parallel lines within A (see Fig. 157) and upon each of the rectangles thus formed to erect a rectangular parallelepiped, as $PQRU$. The area of the upper base RU of $PQRU$ is equal approximately to the area of the intercepted portion RT of S multiplied by the cosine of the angle between the plane of RU (or XY) and the tangent plane at R to S, the accuracy of the approximation being greater and greater as Δx and Δy are taken smaller and smaller. In other words, if τ represents the angle between the tangent plane at R and the XY-plane, we have approximately

$$\text{area } RU = \text{area } RT \quad \cos \tau.$$

Therefore, we have approximately

$$\text{area } RT = \frac{\text{area } RU}{\cos \tau}. \tag{1}$$

If, then, we form the expression (1) corresponding to each one of the parallelepipeds $PQRU$ standing within A and take the sum of the

results, and finally take the limit of this sum as $\Delta x \to 0$ and $\Delta y \to 0$, we shall arrive at an actual expression for S. It remains but to formulate this process analytically.

By Art. 62, the equation of the tangent plane to the given surface $z = f(x, y)$ at R is

$$\frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1) - (z - z_1) = 0, \tag{2}$$

where x_1, y_1, z_1 represent the coördinates of R and where $\frac{\partial z_1}{\partial x_1}$, $\frac{\partial z_1}{\partial y_1}$ represent the values of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at this point. By the familiar formula of solid analytic geometry giving the cosine of the angle between two planes in space,* we find directly that the cosine of the angle τ between the plane (2) and the XY-plane (or $z = 0$) is as follows:

$$\cos \tau = \left[1 + \left(\frac{\partial z_1}{\partial x_1}\right)^2 + \left(\frac{\partial z_1}{\partial y_1}\right)^2\right]^{-\frac{1}{2}}.$$

Hence, dropping the subscripts in order that R may be regarded as having the coördinates (x, y, z), and noting that

$$\textit{area } RU = \Delta y\, \Delta x,$$

we see that (1) takes the form

$$\textit{area } RT = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} \Delta y\, \Delta x. \tag{3}$$

The process of summing all the expressions (3) pertaining to A and finally taking the limit of this sum is evidently analogous to that carried out in the last article with reference to the expression (1) there appearing and it thus is seen to lead to the double integral indicated in the theorem. The student is advised to fill in these details, thus completing the proof.

Just as in the formula for V in the last article the order of integration may be reversed if desired by introducing new limits to the integrals in the manner there described, so likewise the order of integration in the formula for S of the present article may be reversed under the same agreement as to the determination of the new limits of integration.

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* See, for example, the author's "Brief Course in Analytic Geometry" (Holt), p. 250.

Example 1. A right circular cylinder of radius a is cut by a plane which meets the axis of the cylinder at an angle of 45°. Find the area of the cross section thus resulting.

SOLUTION. Take the X-axis along the axis of the cylinder, with origin at the point O where the given plane meets the latter axis. As Y-axis take the line in space which is perpendicular to OX at O and at the same time lies in the given plane. Finally, as Z-axis take the line in space perpendicular to the X- and Y-axes as thus chosen. There results a figure which in the first octant appears as in Fig. 159 and the problem becomes that of determining the shaded area AOB. Four times this area will evidently be the total area sought.

With these axes, the equation of the cylinder becomes

$$y^2 + z^2 = a^2 \tag{4}$$

and the equation of the given plane AOB becomes

$$z = x. \tag{5}$$

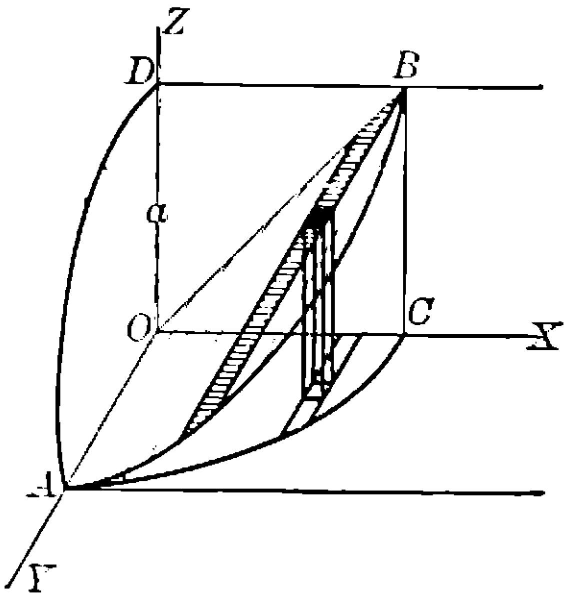


FIG. 159

Moreover, the equation of the curve AC representing the projection of AB upon the XY-plane results, by elementary solid analytic geometry, by eliminating z from (4) and (5) and is thus found at once to be that of the circle

$$x^2 + y^2 = a^2. \tag{6}$$

With (4), (5), and (6) once at hand, we have now but to apply the theorem, noting that, since we are concerned with the area of a portion of the plane (5), the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ involved in the formula for S are to be computed from (5), while the limits of the first or y-integration are to be determined from (6). Thus, we have

$$\begin{aligned} \text{area } AOB &= \int_0^a \int_0^{\sqrt{a^2-x^2}} [1 + 1 + 0]^{\frac{1}{2}}\, dy\, dx = \sqrt{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx \\ &= \sqrt{2}\int_0^a \sqrt{a^2 - x^2}\, dx = \sqrt{2}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}\right]_{x=0}^{x=a} \\ &= \frac{\sqrt{2}}{4}\pi a^2. \end{aligned}$$

The desired area of the entire cross-section is therefore

$$4 \times \frac{\sqrt{2}}{4}\pi a^2 = \sqrt{2}\,\pi a^2.$$

Example 2. In Example 1 find the area of the portion ADB of cylindrical surface intercepted in the first octant between the given plane AOB and the YZ-plane.

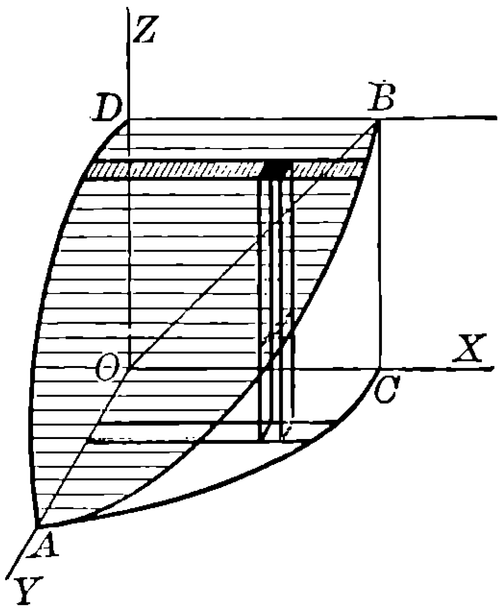


FIG. 160

SOLUTION. Inasmuch as we are now dealing with a portion of the surface of the cylinder, the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ to be used in applying the theorem are to be determined from (4). They are, therefore, as follows:

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = -\frac{y}{z} = -\frac{y}{\sqrt{a^2 - y^2}}.$$

Moreover, it is convenient in the present instance (for reasons to be seen presently) to integrate first with respect to x, in which case the x-limits, as determined from (6) (see Fig. 160 also), are 0 and $\sqrt{a^2 - y^2}$. Application of the theorem thus gives

$$\text{area } ADB = \int_0^a \int_0^{\sqrt{a^2-y^2}} \left[1 + 0 + \frac{y^2}{a^2 - y^2}\right]^{\frac{1}{2}} dx\, dy = a \int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{dx\, dy}{\sqrt{a^2 - y^2}}$$

$$= a \int_0^a dy = a^2.$$

EXERCISES

1. Work illustrative Example 1 of Art. 152 in case the intersecting plane is $z = mx$ instead of $z = x$, other conditions remaining the same.

Ans. $\frac{\pi a^2}{m}(1 + m^2)^{\frac{1}{2}}$.

2. Work illustrative Example 2 of Art. 152 in case the intersecting plane is $z = mx$ instead of $z = x$, other conditions remaining the same. *Ans.* $\frac{a^2}{m}$.

3. Work illustrative Example 1 of Art. 152 in case the cylinder used is the elliptic cylinder $b^2y^2 + a^2z^2 = a^2b^2$ instead of the circular cylinder $y^2 + z^2 = a^2$, other conditions remaining the same. *Ans.* $\sqrt{2}\,\pi ab$.

4. Work illustrative Example 2 of Art. 152 in case the surface used is the sphere $x^2 + y^2 + z^2 = a^2$ instead of the cylinder there employed and the intersecting plane used is $y = mx$. *Ans.* a^2 arc cot m.

5. Find the area of that part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ which is intercepted by the coördinate planes. *Ans.* $\frac{1}{2}(b^2c^2 + c^2a^2 + a^2b^2)^{\frac{1}{2}}$.

6. Show by means of the theorem of Art. 152 that the area of a sphere of radius r is $4\pi r^2$.

7. The center of a sphere of radius a lies on the surface of a right circular cylinder of radius $\frac{a}{2}$. Find the surface of the cylinder intercepted by the sphere. *Ans.* $4a^2$.

HINT. Take axes as in Fig. 161. Then one fourth of the desired area is $AHOCFA$ and its projection on the XY-plane is $AEOCBA$. The equations of the sphere and of the cylinder become respectively

$$x^2 + y^2 + z^2 = a^2 \quad \text{and} \quad y^2 + z^2 = ay$$

and, upon eliminating z from these equations, we obtain as the equation of the curve ABC the parabola

$$x^2 + ay = a^2.$$

Integrate first with respect to x.

8. In Ex. 7 determine the surface of the sphere intercepted by the cylinder. *Ans.* $2(\pi - 2)a^2$.

HINT. Using Fig. 161, one fourth of the desired area is $AFCBA$, the projection of which on the YZ-plane is the semicircle AHO. In order that this projection may be on the XY-plane, as required in the theorem of Art. 161, interchange the positions of the X- and Z-axes. The equations of the sphere and of the cylinder then become respectively

$$x^2 + y^2 + z^2 = a^2 \quad \text{and} \quad y^2 + x^2 = ay.$$

The result comes to depend upon the value of the integral

$$\int_0^a \arcsin \sqrt{\frac{y}{y + a}}\, dy.$$

To integrate, substitute $y = a \tan^2 \theta$.

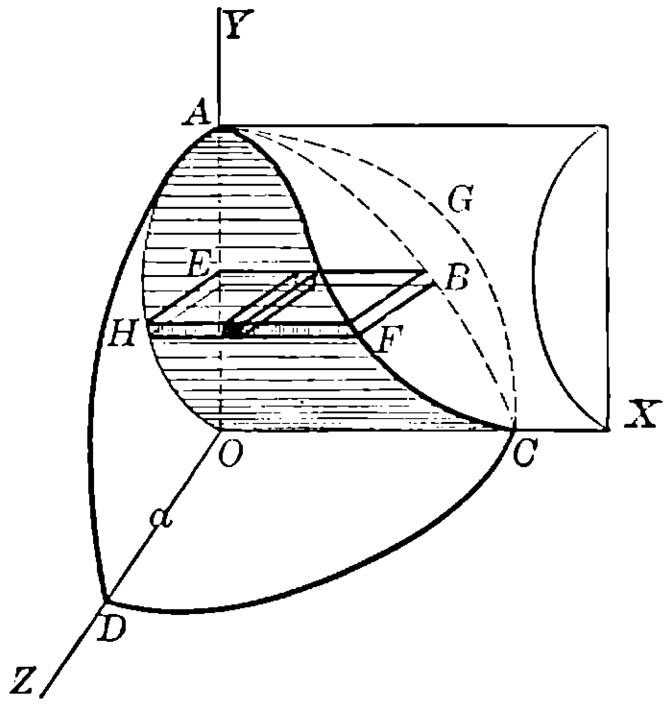


FIG. 161

9. Find the area of that portion of the sphere $x^2 + y^2 + z^2 = 2\,az$ which is intercepted by one nappe of the cone $x^2 + y^2 = z^2$. *Ans.* $2\,\pi a^2$.

10. Find the area of the portion of the surface of the cylinder $y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is contained in the cylinder $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{24}{5}a^2$.

11. Find the surface of the cylinder $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is included in the sphere $x^2 + y^2 + z^2 = a^2$. *Ans.* $\frac{3}{2}\sqrt{3}\,\pi a^2$.

12. Formulate as integrals, without undertaking their evaluation, answers to the following questions:

(*a*) A hole of square cross-section, length of side l, is cut through a sphere of radius a, the axis of the hole lying along a diameter. How much of the spherical area is intercepted?

(*b*) A round hole of radius a is bored diametrically across a log of radius b. How much of the area is intercepted?

(*c*) Answer question (*b*) when the hole is square in cross-section, length l on a side, other conditions remaining the same.

153. Volumes as determined by triple integration. As an application to geometry of the idea of the triple integral defined in Art. 147, we shall now show how such an integral may be used, if desired, in solving the fundamental problem of Art. 150; namely, the problem of finding the volume under a given surface. For brevity, we shall confine ourselves to illustrations rather than attempt to state a general theorem, the latter being, however, readily done.

Example 1. Find by triple integration the volume of that portion of the sphere

$$x^2 + y^2 + z^2 = a^2$$

which lies in the first octant.

SOLUTION. Imagine the volume in question to be decomposed into a large number of rectangular parallelepipeds (and parts of parallelepipeds) each of dimensions Δz, Δy, Δx, as indicated in Fig. 162. Sum up first all the (complete) parallelepipeds lying within a column QP of cross-section $\Delta y\ \Delta x$ and extending from the XY-plane to the value of z at P; that is, from $z = 0$ to $z = \sqrt{a^2 - x^2 - y^2}$, as obtained from (1). Next, having carried out this process for each column QP standing on the strip $IFJK$ of the XY-plane, noting that along this strip y varies from 0 to the value $y = \sqrt{a^2 - x^2}$ upon the circle AB, take the sum of all these results, thus arriving at the sum of all the parallelepipeds in the slice $HFILJK$ of the solid. Now, sum up all these last results corresponding to the different slices, it being observed that in this process x varies from $x = 0$ to $x = a$. The limit as $\Delta z \to 0$, $\Delta y \to 0$, $\Delta x \to 0$ of this last result, which (limit) may evidently be regarded as the actual volume of the solid, becomes, upon continued application of the theorem of Art. 138, identified with the triple integral

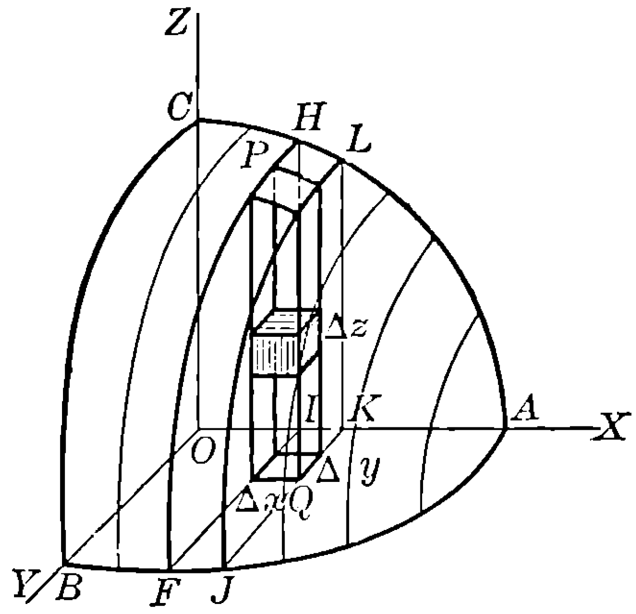


FIG. 162

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

Hence, it remains but to evaluate this integral.

We have

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = \int_0^a\int_0^{\sqrt{a^2-x^2}}(a^2 - x^2 - y^2)^{\frac{1}{2}}\,dy\,dx$$

$$= \frac{\pi}{4}\int_0^a (a^2 - x^2)\,dx = \tfrac{1}{6}\,\pi a^3.$$

The volume of the entire sphere is therefore equal to eight times this result, or $\frac{4}{3}\,\pi a^3$.

Example 2. Two right cylinders having the same radius a meet each other in such a way that the axes of the two are at right angles. Find by triple integration the volume common to the two cylinders.

SOLUTION. Take the Y-axis and the Z-axis along the axes of the two cylinders. Then the equations of the cylinders become respectively

$$x^2 + z^2 = a^2 \qquad (1)$$

and

$$x^2 + y^2 = a^2. \qquad (2)$$

Confining the attention to the common volume lying in the first octant, which evidently is one eighth of the whole, and noting that the arc AB is that of a circle whose equation is (2), we find by the reasoning similar to that in Ex. 1 that

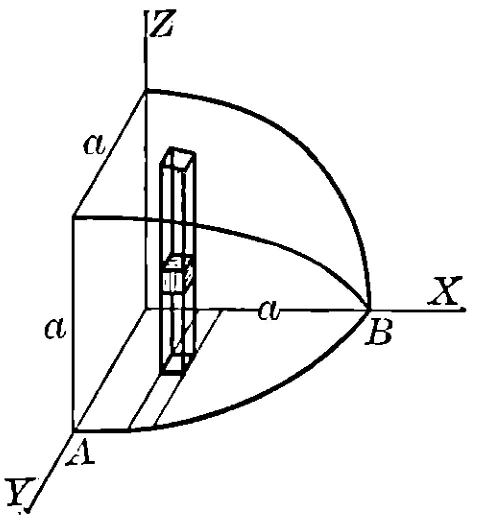


FIG. 163

$$\frac{V}{8} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\, dy\, dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy\, dx$$

$$= \int_0^a (a^2 - x^2)\, dx = \frac{\quad}{3} a^3.$$

The entire volume sought is therefore $\quad a^3$.

Note. Whenever the limits of integration in a triple integral are supplied, as illustrated in the foregoing examples, in such a way that all the elements of volume $\Delta z\, \Delta y\, \Delta x$ pertaining to a given volume V are included in the summation which gives rise to this integral, we say that the limits have been supplied so as to **cover** the entire volume V (compare Art. 151).

EXERCISES

Work the following examples by means of triple integration. Compare Exs. page 298.

1. Find the volume of the tetrahedron bounded by the coördinate planes and the plane.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \qquad \textit{Ans.}\ \tfrac{1}{6}\, abc.$$

2. Find the volume of the solid contained between the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2\,ax$, and the XY-plane. *Ans.* $\frac{3}{2}\pi a^3$.

3. Find the volume of the cylindrical column standing upon the area common to the two parabolas $y = x^2$ and $x = y^2$ as a base and intercepted by the surface $z = 4 + y - x^2$.

4. Prove that the entire volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\frac{4}{3}\pi abc$.

II. APPLICATIONS TO PHYSICS

154. Definitions. The **mass** of a body may be regarded as that property of it by virtue of which it has weight. Moreover, if the mass be doubled, the weight will be doubled; if the mass be halved, the weight will be halved, etc. The common (engineering) unit of mass is called the **pound-mass,** this being the amount of mass present when the gravitational attraction upon it amounts to one pound of force.

Suppose a mass M to be given, and that it occupies a volume V Take any small portion, ΔV, of this volume and suppose that the mass within ΔV is ΔM. Form the ratio

$$\frac{\Delta M}{\Delta V}. \tag{1}$$

This ratio is known as the **average density** of the portion ΔM of M.

Suppose next that a certain definite point P is selected within ΔV and that the limiting value of the ratio (1) is then determined as ΔV (and hence ΔM) approaches zero in such a way that P always remains within ΔV The value of this limiting ratio, which is usually indicated by ρ, is called the **density of *M* at *P*.** That is, we have by definition

$$\rho = \textit{density at } P = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}. \tag{2}$$

If the value of ρ at every point P within M is the *same*, M is said to be **homogeneous;** if not, M is called **heterogeneous.** In what follows we shall be dealing with homogeneous masses only.

We observe finally that if a homogeneous mass M of density ρ occupies a volume V then there exists the relation

$$M = \rho V \tag{3}$$

In fact, if this were not the case, it could be shown that the value of the limit appearing on the right in (2) would have to be different from ρ at at least some point P in M, thus contradicting the hypothesis that M is homogeneous. For brevity, we omit the logical details.

It is frequently convenient to look upon an individual point as endowed with mass. A point thus regarded is called a **material particle,** or simply a **particle.**

155. Center of gravity of a system of particles. Suppose that a system of n particles is given and suppose for simplicity that they all lie in one plane. Let m_1, m_2, m_n be respectively the masses of

the particles and, having taken a set of axes anywhere in the plane, suppose that the coördinates of $m_1, m_2, m_3,$ m_n are respectively $(x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$. Then the products

$$m_1x_1, m_2x_2, m_3x_3, \quad m_nx_n$$

are called respectively the **moments** of the particles $m_1, m_2, \cdots, m_n$ with respect to the Y-axis. Similarly, the products $m_1y_1,$ $m_2y_2,$ m_ny_n are called the moments with respect to the X-axis.

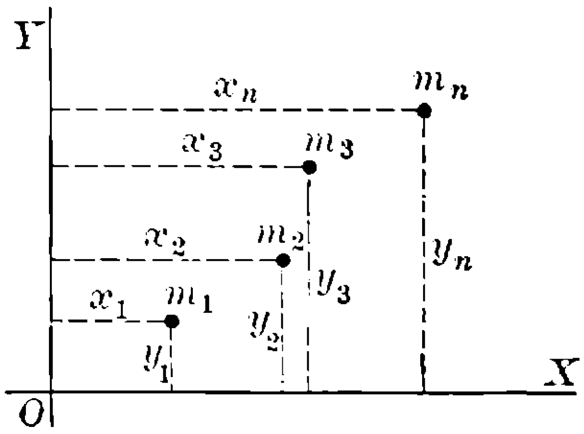


FIG. 164

The **center of gravity** of the system $m_1, m_2,$ m_n is defined to be that point $(\bar{x}, \bar{y})$ at which, if all the masses $m_1, m_2,$ $\cdot, m_n,$ were concentrated, the moments of the resulting (hypothetical) particle with respect to the X- and Y-axes would be equal respectively to the sum of the moments of the actual particles with respect to these axes; that is, $\bar{x}$ and $\bar{y}$ are defined as follows:

$$(m_1 + m_2 + \quad + m_n)\bar{x} = m_1x_1 + m_2x_2 + \quad + m_nx_n,$$

$$(m_1 + m_2 + \quad + m_n)\bar{y} = m_1y_1 + m_2y_2 + \quad + m_ny_n.$$

Solving these relations for $\bar{x}$ and $\bar{y}$, we obtain the following formulas for the location of the center of gravity of the system:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \quad + m_nx_n}{m_1 + m_2 + \quad + m_n}, \quad \bar{y} = \frac{m_1y_1 + m_2y_2 + \quad + m_ny_n}{m_1 + m_2 + \quad + m_n}.$$

The physical significance of the center of gravity may be described as follows: Suppose that the various particles lie in a vertical plane and are rigidly joined to each other and to the center of gravity by means of weightless wires. Then the entire system, if suspended upon this center, will remain at rest when acted upon by the force of gravity; that is, there will be no rotational or other effect upon the system, or as commonly expressed, the system will be "perfectly balanced."

Example. Find the center of gravity of a system of four particles lying in a vertical plane if the masses of the particles are respectively 2, 3, 4, and 5 ounces and their locations, as referred to coördinate axes in the plane, are respectively (1, 2), (2, 1), (3, 2), and (3, 3).

SOLUTION. By the foregoing formula, the coördinates $(\bar{x}, \bar{y})$ of the center of gravity are

$$\bar{x} = \frac{2 \quad 1 + 3 \quad 2 + 4 \cdot 3 + 5 \quad 3}{2 + 3 + 4 + 5} = \frac{35}{14} = 2\tfrac{1}{2},$$

$$\bar{y} = \frac{2 \cdot 2 + 3 \quad 1 + 4 \quad 2 + 5 \quad 3}{2 + 3 + 4 + 5} = \frac{30}{14} = 2\tfrac{1}{7}.$$

156. Center of gravity of a plane area. Suppose that a closed area A is given in a plane which we shall regard as the XY-plane, and suppose that this area is endowed with mass of uniform density ρ, as is realized, for example, very nearly in a thin sheet of metal such as tin-foil. A slight extension of the reasoning of the last article enables us to determine the center of gravity of this area as follows:

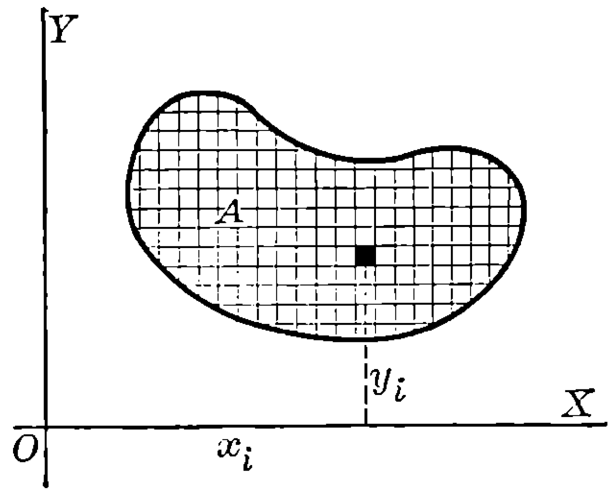


FIG. 165

Theorem. *The coördinates* $(\bar{x}, \bar{y})$ *of the center of gravity of a plane area* A *are given by the formulas*

$$\bar{x} = \frac{\displaystyle\iint x\, dy\, dx}{A}, \qquad \bar{y} = \frac{\displaystyle\iint y\, dy\, dx}{A}, \tag{1}$$

where the limits for the double integrals are to be so supplied that the integration shall cover the entire area A.

PROOF. Draw a set of lines parallel to the Y-axis at equal small distances Δx apart and draw a similar set of parallels to the X-axis at equal small distances Δy apart (compare Art. 148). The area A thus becomes decomposed into a large number of small rectangles and parts of rectangles, each endowed with mass and hence in the nature approximately of a particle. The whole thus forms, approximately, a system of particles. The center of gravity of this system, as determined by the formula of Art. 155, naturally furnishes a good approximation to the values of $\bar{x}$ and $\bar{y}$ belonging to A. Moreover, the limits as $\Delta x \to 0$ and $\Delta y \to 0$ of these approximate values of $\bar{x}$ and $\bar{y}$ may be regarded as the correct values of these quantities. It remains, then, but to formulate this process analytically.

Consider any one of the rectangles, say the ith. Its area is $\Delta y\, \Delta x$ and therefore its mass, by (3) of Art. 154, is $\rho\, \Delta y\, \Delta x$. Hence, if (x_i, y_i) be the coördinates of any selected point within this rectangle, the moments of the mass of the rectangle with respect to the X- and Y-axes are approximately

$$x_i \rho\, \Delta y\, \Delta x \quad \text{and} \quad y_i \rho\, \Delta y\, \Delta x.$$

Therefore, the formula of the last article as applied to the system of masses under consideration gives the following as the approximate values of $\bar{x}$ and $\bar{y}$ pertaining to A

$$\bar{x} = \frac{\Sigma x_i \rho \,\Delta y\, \Delta x}{\Sigma \rho \,\Delta y\, \Delta x}, \quad \bar{y} = \frac{\Sigma y_i \rho\, \Delta y\, \Delta x}{\Sigma \rho\, \Delta y\, \Delta x}, \tag{2}$$

where the symbol Σ (Greek letter sigma) placed before any expression (such as $x_i \rho \,\Delta y\, \Delta x$) indicates the *sum* of all such expressions pertaining to A. If now in (2) we allow first Δy to approach zero and then Δx to approach zero, the various sums in question are seen, by reasoning analogous to that of Art. 150, to become in the limit double integrals. In fact, we thus obtain as the exact expressions for $\bar{x}$ and $\bar{y}$ the following:

$$\bar{x} = \frac{\int\int x\rho \,dy\,dx}{\int\int \rho \,dy\,dx}, \quad \bar{y} = \frac{\int\int y\rho \,dy\,dx}{\int\int \rho \,dy\,dx}, \tag{3}$$

where the limits for the double integrals are to be so supplied that the integration shall cover the entire area A.

We now note that the factor ρ, being a constant, may be removed to the outside of each double integral in (3), and this having been done, it cancels from numerator and denominator. Moreover, there then remains in the denominator a double integral which, being the limit of the sum of all the rectangular areas $\Delta y\, \Delta x$ throughout A, is equal to A itself.

Thus the expressions (3) reduce to (1) and the proof is complete.

Example. Find the coördinates of the center of gravity of the portion of the parabola $y^2 = 4 - x$ which lies in the first quadrant.

SOLUTION. For the area A in question we have

$$A = \int_0^4 \int_0^{\sqrt{4-x}} dy\,dx = \int_0^4 \sqrt{4-x}\,dx$$
$$= -\frac{2}{3}\Big[(4-x)^{\frac{3}{2}}\Big]_{x=0}^{x=4} = \frac{16}{3}. \tag{4}$$

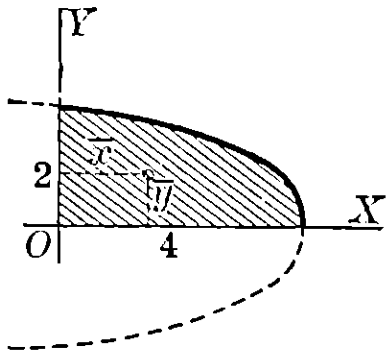


FIG. 166

For the numerator of the expression for $\bar{x}$ (see formula), we have

$$\int_0^4 \int_0^{\sqrt{4-x}} x\,dy\,dx = \int_0^4 x\sqrt{4-x}\,dx = -\frac{2}{15}\Big[(8+3x)(4-x)^{\frac{3}{2}}\Big]_{x=0}^{x=4} = \frac{128}{15}. \tag{5}$$

For the numerator of the expression for $\bar{y}$, we have

$$\int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx = \frac{1}{2}\int_0^4 (4-x)\,dx = \frac{1}{2}\Big[4x - \frac{x^2}{2}\Big]_{x=0}^{x=4} = 4. \tag{6}$$

From (4), (5), and (6) we have as the desired values of $\bar{x}$ and $\bar{y}$ (see formula)

$$\bar{x} = \frac{\frac{128}{15}}{\frac{16}{3}} = \frac{8}{5}, \qquad \bar{y} = \frac{4}{\frac{16}{3}} = \frac{3}{4}.$$

157. Center of gravity of volumes. In case the system of particles considered in Art. 155 does not all lie in one plane, the $\bar{x}$ and the $\bar{y}$ of the center of gravity, as referred to a set of X-, Y-, and Z-axes in space, continue to be given by the formulas of that article, it being understood that $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ represent as before the x- and the y-coördinates respectively of the various particles. Besides the formulas for $\bar{x}$ and $\bar{y}$, there is now an analogous formula for $\bar{z}$ wherein the z-coördinates $z_1, z_2, \cdots, z_n$ of the various particles occur. This being premised, the reasoning of the last article may be extended directly to the determination of formulas for the coördinates $(\bar{x}, \bar{y}, \bar{z})$ of the center of gravity of any volume V throughout which there is a mass of uniform density. The result, proof of which should be supplied by the student, is as follows:

$$\bar{x} = \frac{\iiint x\,dz\,dy\,dx}{V}, \quad \bar{y} = \frac{\iiint y\,dz\,dy\,dx}{V}, \quad \bar{z} = \frac{\iiint z\,dz\,dy\,dx}{V},$$

where the limits for the triple integrals are to be so supplied that the integration shall cover the entire volume V

158. General remarks. In finding the centers of gravity of areas or volumes the following general principles, which are self-evident physically and which can be deduced analytically from the results in Arts. 155–157, are often useful:

(1) Any plane or line of symmetry in a figure must contain the center of gravity.

(2) If an area or a volume consists of several portions for each of which the center of gravity can be found, then the mass of each portion may be imagined as concentrated at its own center of gravity, thus reducing the problem to the consideration of a set of particles.

EXERCISES

1. Find the center of gravity of each of the following systems of particles:

(*a*) Three particles of masses 1 ounce, 2 ounces, and 3 ounces situated respectively at the points (2, 3), (4, 1), (3, 5). *Ans.* $\bar{x} = 3\frac{1}{6}$, $\bar{y} = 3\frac{1}{3}$.

(*b*) Three particles of equal masses situated at the points mentioned in (*a*). *Ans.* $\bar{x} = 3$, $\bar{y} = 3$.

(*c*) Three particles whose masses are as described in Ex. 1 situated respectively at the points (1, 2, 3), (2, 4, 1), (0, 3, 5). *Ans.* $\bar{x} = \frac{5}{6}$, $\bar{y} = 3\frac{1}{6}$, $\bar{z} = 3\frac{1}{3}$.

(*d*) Four equal particles situated

(1) at the points (1, 2), (3, 4), (2, 0), (4, 1). *Ans.* $\bar{x} = 2\frac{1}{2}$, $\bar{y} = 1\frac{3}{4}$.

(2) at the points (0, 1, 2), (1, 3, 4), (2, 2, 0), (3, 4, 1). *Ans.* $\bar{x} = 1\frac{1}{2}$, $\bar{y} = 2\frac{1}{2}$, $\bar{z} = 1\frac{3}{4}$.

2. Find the center of gravity of each of the following areas. Make use wherever possible of the remarks in Art. 158:

(*a*) The entire parabolic segment (above and below the X-axis) shown in Fig. 166; *Ans.* $\bar{x} = \frac{8}{5}$, $\bar{y} = 0$.

(*b*) A semicircle of radius a; *Ans.* $\bar{x} = \frac{4a}{3\pi}$.

(*c*) A quadrant of a circle of radius a; *Ans.* $\bar{x} = \bar{y} = \frac{4a}{3\pi}$.

(*d*) The arch of the sinusoid $y = \sin x$ lying between $x = 0$ and $x = \pi$;

Ans. $\bar{x} = \frac{\pi}{2}$, $\bar{y} = \frac{\pi}{8}$.

(*e*) The area bounded by the parabolas $y^2 = 4ax$, $x^2 = 4ay$;

Ans. $\bar{x} = \bar{y} = \frac{9}{5}a$.

(*f*) A quadrant of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Ans. $\bar{x} = \bar{y} = \frac{256a}{315\pi}$.

(*g*) Work Ex. (*f*) when the equation of the curve is taken in the parametric form $x = a\cos^3\phi$, $y = a\sin^3\phi$;

(*h*) One arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$;

Ans. $\bar{x} = \pi a$, $\bar{y} = \frac{5}{6}a$.

(*i*) The area between the portion of the circle $x^2 + y^2 = a^2$ lying in the first quadrant and the tangents to its extremities; *Ans.* $\bar{x} = \bar{y} = \frac{2a}{12 - 3\pi}$.

(*j*) The area between the portion of the circle $x^2 + y^2 = a^2$ lying in the first quadrant and the chord joining its extremities; *Ans.* $\bar{x} = \bar{y} = \frac{2a}{3\pi - 6}$.

(*k*) The portion which lies in the first quadrant of the circular ring bounded by the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$ $(b < a)$;

Ans. $\bar{x} = \bar{y} = \frac{4}{3\pi}\,\frac{a^2 + ab + b^2}{a + b}$

(*l*) A square of side a having a semicircle erected upon one side;

(*m*) A square of side a, having semicircles erected upon two adjacent sides.

3. Find the center of gravity of each of the following volumes:

(*a*) The hemisphere formed by the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying to the right of the YZ-plane. *Ans.* $\bar{x} = \frac{3}{8}a$, $\bar{y} = \bar{z} = 0$.

(*b*) The first octant of the sphere $x^2 + y^2 + z^2 = a^2$.

Ans. $\bar{x} = \bar{y} = \bar{z} = \frac{3}{8}a$.

(*c*) The solid obtained by revolving about the X-axis the area bounded by the upper branch of the parabola $y^2 = 2px$, the X-axis, and the line $x = a$.

Ans. $\bar{x} = \frac{2}{3}a$, $\bar{y} = \bar{z} = 0$.

(*d*) The solid obtained by revolving about the X-axis the area bounded by the hyperbola $x^2 - y^2 = a^2$, the X-axis, and the line $x = 2a$.

Ans. $\bar{x} = \frac{27}{16}a$, $\bar{y} = \bar{z} = 0$.

(*e*) The volume obtained by revolving about the X-axis a quadrant of the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\bar{x} = \frac{21}{128}a$, $\bar{y} = \bar{z} = 0$.

4. Prove that the center of gravity of any triangular area lies at the intersection of the medians.

5. Prove that the center of gravity of a regular tetrahedron lies at the intersection of the altitudes.

6. Prove that the center of gravity of a right circular cone of altitude h lies on the axis at a distance $\frac{1}{4}h$ from the base.

7. Find the center of gravity of a volume composed of a right circular cylinder of radius a and altitude h surmounted by a right circular cone of altitude c and base-radius a.

8. Find the center of gravity of the volume remaining when a right circular cone of base-radius a and altitude c is removed from a right circular cylinder having the same base and altitude.

159. Moment of inertia. If a particle of mass m be situated at the distance r from a fixed line, then the product mr^2 is called the **moment of inertia** of m about this line as axis. If, in particular, the coördinates of m are (x, y), the moment of inertia I of m about an axis through the origin perpendicular to the XY-plane is given by the formula

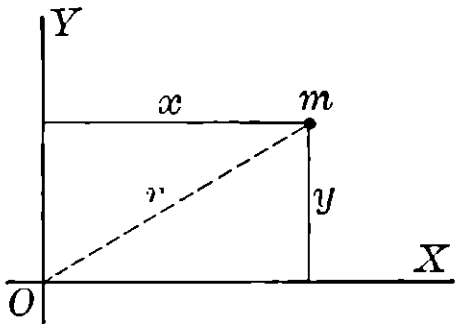


FIG. 167

$$I = m(x^2 + y^2). \qquad (1)$$

It follows that if a system of n particles is given having the masses $m_1, m_2, \cdots, m_n$ and situated respectively at the points (x_1, y_1), (x_2, y_2), (x_n, y_n), then the moment of inertia I of the system about an axis through the origin perpendicular to the XY-plane will be equal to the sum of all the expressions (1) pertaining to the system; that is, using the usual symbols to express this idea, we shall have

$$I = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2).$$

The moment of inertia of any plane slab, or lamina, about an axis is now readily formulated as follows:

Theorem. *The moment of inertia I of a plane lamina of area A, thickness k, and density ρ is determined by the formula*

$$I = \rho k \int\int (x^2 + y^2)\, dy\, dx,$$

where the limits for the double integral are to be so supplied that the integration shall cover the entire area A, and where it is understood that I as thus determined is about the line through the origin perpendicular to the XY-plane as axis.

The proof of this theorem, being closely analogous to that in Art. 156, is left to the student with the aid of Fig. 168. He should also note that if, instead of a lamina of thickness k, a plane area of no thickness but endowed with mass of density ρ be used, the formula for I is as stated in the theorem except that the factor k is omitted.

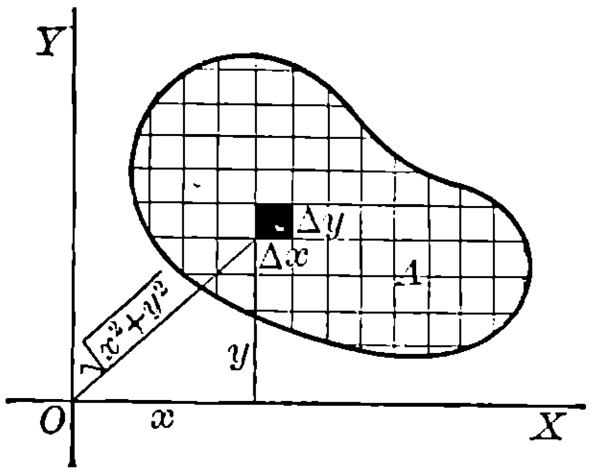


FIG. 168

Example. Find the moment of inertia of a circular disk of radius a, thickness k, and density ρ about the perpendicular through the center as axis.

SOLUTION. Using Fig. 169, the foregoing theorem gives

$$I = \rho k \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx = 2\rho k \int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx + \tfrac{2}{3}\rho k \int_{-a}^{a} (a^2-x^2)^{\frac{3}{2}}\,dx$$

$$= \tfrac{1}{4}\pi\rho k a^4 + \tfrac{1}{4}\pi\rho k a^4 = \tfrac{1}{2}\pi\rho k a^4. \quad (2)$$

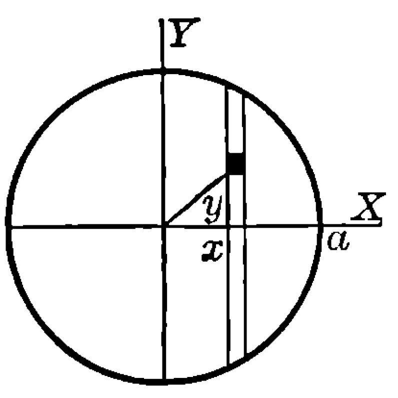


FIG. 169

This result, however, is most conveniently expressed in another form. Thus, the mass M of the disk is equal to its area πa^2 multiplied by its thickness k multiplied by the density ρ; that is,

$$M = \pi a^2 k\rho,$$

and hence, taking out the factor $\pi a^2 k\rho$ from the second member of (1), the final result takes the form

$$I = \tfrac{1}{2} Ma^2. \quad (3)$$

160. Physical significance of moment of inertia. An adequate description of the rôle played by the moment of inertia in physics lies beyond the scope of the present text, but the following important fact may be cited, proof of which belongs to the science of dynamics.

If a heavy body such as a flywheel be rotating freely about an axis with angular velocity ω, then the work (see Art. 145 (c)) which the wheel is capable of doing before being brought to rest is given by the formula

$$W = \tfrac{1}{2} I\omega^2,$$

where I is the moment of inertia of the body. The work W stored up in a moving body in this manner is known as its **kinetic energy.**

If the mass of the rotating body is measured in pounds and its dimensions are measured in feet, while the angular velocity ω is measured in radians per second, then the value of W as determined by the foregoing formula is measured in terms of the unit of work known in physics as the "foot-poundal." In order to convert

W as thus measured into its corresponding measure in foot-pounds (see Art. 145 (c)) it suffices merely to divide it by $g = 32$ (approximately), g being the familiar constant met with in the study of the laws of a falling body (see Ex. 3, page 109).

Example. A small heavy weight of mass m pounds is suspended h feet from the ground by means of a string which is wound upon a weightless spool attached on the same shaft with a heavy circular disk of radius a feet, thickness k feet, and density ρ pounds per cubic foot. If the weight be allowed to descend, how fast will the disk be revolving when the weight strikes the ground?

SOLUTION. Using the value for the moment of inertia for the disk as obtained in (3) of the last article and noting that the work done by the weight in descending the distance h is, by Art. 145 (c), mh foot-pounds, or mhg foot-poundals, the foregoing formula gives the following equation for determining the value of ω at the instant in question:

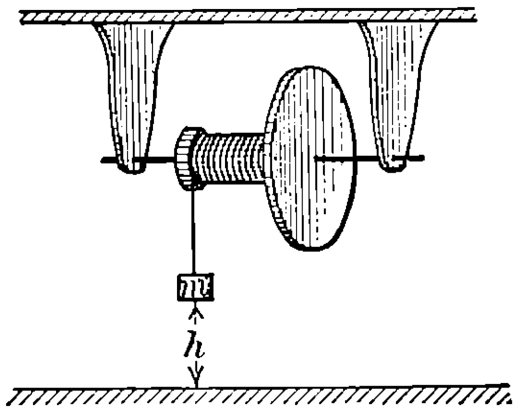


FIG. 170

$$mhg = \tfrac{1}{2}(\tfrac{1}{2}\,Ma^2)\omega^2,$$

where M is the mass of the disk. It follows that

$$\omega = \frac{2}{a}\left(\frac{mhg}{M}\right)^{\frac{1}{2}} \text{radians per second.}$$

It will be noted that the answer is independent of the radius of the spool.

161. Radius of gyration. Suppose that the moment of inertia of a certain mass M with respect to a given axis L is I. Then, the quantity R determined by the equation

$$I = MR^2$$

is known as the **radius of gyration** of M about L. As thus defined, the radius of gyration is seen to be that distance from L at which, if the entire mass M were concentrated, the moment of inertia of the original mass would remain unchanged.

Example. Find the radius of gyration R of the circular disk described in the Example of Art. 159.

SOLUTION. Having obtained the moment of inertia in the form (3) of Art. 159, we have at once

$$R^2 = \frac{a^2}{2}.$$

Hence,

$$R = \frac{a}{\sqrt{2}} = \frac{1}{2}a\sqrt{2}.$$

Note. As an illustration of the physical significance of the radius of gyration, the following may be cited. If a baseball bat be swung round from the end of the handle as a pivot, the radius of gyration determines the distance from the end of the handle to the point at which it would be most advantageous to hit the ball, this point being known as the **center of percussion.** Since the bat is ordinarily swung from the shoulder instead of the end of the handle, the center of percussion lies somewhat farther out in practice than just indicated.

EXERCISES

1. Find the moment of inertia of each of the following systems of particles about an axis through the origin perpendicular to their plane:

(*a*) Three particles of masses 1, 2, and 3 units situated respectively at the points (2, 3), (4, 1), (3, 5); *Ans.* 149.

(*b*) Four particles of masses 1, 3, 2, 1 units situated respectively at the points (0, 2), (3, 1), (2, 2), (4, 0). *Ans.* 67.

2. Determine the radius of gyration of each of the systems of particles described in Ex. 1. *Ans.* (*a*) $\sqrt{25\frac{5}{6}}$; (*b*) $\sqrt{9\frac{4}{7}}$.

3. Assuming that the unit of mass is 1 pound, find the kinetic energy of each of the systems of particles described in Ex. 1 when rotating about the given axis at the rate of 2 radians per second.

Ans. (*a*) $9\frac{5}{16}$ ft.-lbs.; (*b*) $4\frac{1}{8}$ ft.-lbs. (approx.).

4. Find in terms of the mass M the moment of inertia I of homogeneous plane laminas (or areas) having the respective forms indicated below. From your results state the value of the radius of gyration in each instance:

(*a*) A right triangle of sides a, b about the vertex at the right angle;

Ans. $\frac{1}{6}(a^2 + b^2)M$.

(*b*) A right triangle of sides a, b about the vertex at the intersection of side a with the hypotenuse; *Ans.* $\left(\frac{a^2}{2} + \frac{b^2}{6}\right)M$.

(*c*) A rectangle of dimensions a, b about its center;

Ans. $\frac{1}{12}(a^2 + b^2)M$.

(*d*) A rectangle of sides a, b about one vertex;

Ans. $\frac{1}{3}(a^2 + b^2)M$.

(*e*) An equilateral triangle of side a about the intersection of its altitudes.

5. Find the moment of inertia about the origin of the portion of the parabola $y^2 = 4 - x$ which lies in the first quadrant. See Example, Art. 156.

Ans. $\frac{156}{35}M$.

6. Show that the moment of inertia of a straight rod or wire of length l about a perpendicular through one end is given by the formula

$$I = \tfrac{1}{3}Ml^2.$$

7. Having given a plane area A of density ρ situated in the XY-plane, show that its moments of inertia about the X-axis and the Y-axis are given respectively by the formulas

$$I_x = \rho \int \int y^2 \, dy \, dx, \qquad I_y = \rho \int \int x^2 \, dy \, dx,$$

where the limits for the double integrals are to be so supplied that the integration shall cover the entire area A. Compare Art. 159.

8. By means of the formulas in Ex. 7 find the moment of inertia of each of the following homogeneous mass-areas, M being the mass; also state the radius of gyration in each case:

(*a*) A right triangle of sides a, b about the side a; *Ans.* $\frac{1}{6} Mb^2$.

(*b*) An isosceles triangle of altitude h about its base; *Ans.* $\frac{1}{6} Mh^2$.

(*c*) A circle of radius a about a diameter; *Ans.* $\frac{1}{4} Ma^2$.

(*d*) The area cut off from the parabola $y^2 = 4\,px$ by the line $x = p$ and lying above the X-axis about (1) the X-axis; (2) the Y-axis;

Ans. $I_x = \frac{4}{5} p^2 M$; $I_y = \frac{3}{7} p^2 M$.

(*e*) The area under one arch of the curve $y = \sin x$ about its base. See Ex. 3, page 250.

9. By means of the result for Ex. 8 (*c*) find the kinetic energy of a circular disk of 6-inch radius and mass 2 pounds when revolved at the rate of 2 revolutions per second about its diameter.

10. How far would the revolving disk mentioned in Ex. 9 be capable of lifting vertically a 3-pound weight before coming to rest?

11. If I_x, I_y, and I_0 represent respectively the moments of inertia of an area A about the X-axis, the Y-axis, and the line through the origin perpendicular to these axes, show that there exists the following relation (See Ex. 7):

$$I_0 = I_x + I_y.$$

12. If an area A lying in the XY-plane be revolved about the X-axis so as to form a solid of revolution of density ρ, show that the moment of inertia of the solid about the X-axis is determined by the formula

$$I_x = 2\,\pi\rho \int \int y^3 \, dy \, dx,$$

where the limits for the double integral are to be so supplied that the integration shall cover the entire area A.

13. By means of the formula in Ex. 12 find the moment of inertia of each of the following homogeneous mass-volumes, M being the mass:

(*a*) A cylinder of revolution of radius a about its axis. *Ans.* $\frac{1}{2} Ma^2$.

(*b*) A circular cone whose base is of radius a about its axis. *Ans.* $\frac{3}{10} Ma^2$.

(*c*) A sphere of radius a about a diameter. *Ans.* $\frac{2}{5} Ma^2$.

(*d*) The solid obtained by revolving about the X-axis the area bounded by the parabola $y^2 = 4\,px$ and the line $x = p$.

14. Work the Example of Art. 160 when, instead of the circular disk there mentioned, a sphere of radius a feet and mass M pounds is used, other conditions remaining as before. See Ex. 13 (*c*).

CHAPTER XVII

DIFFERENTIAL EQUATIONS

162. Definitions. An equation containing derivatives or differentials is called a **differential equation.**

For example,

$$\frac{d^2y}{dx^2} + y = 0, \tag{1}$$

$$(1 + y)\,dx - (1 - x)\,dy = 0, \tag{2}$$

$$\frac{ds}{dt} + s\cos t = \sin 2t, \tag{3}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u. \tag{4}$$

Equations of this character have already been encountered in the earlier chapters (see, for example, page 126), but in those instances they played the rôle merely of equations which were satisfied identically by *given* functions. In distinction to this, we are to consider in the present chapter the converse problem; namely, a differential equation being given, to *determine* the function, or functions, which satisfy it identically. Any function thus related to a given differential equation is called a **solution** of the equation.

For example, the function

$$y = c_1 \sin x, \tag{5}$$

where c_1 is an arbitrary constant, is a solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0, \tag{6}$$

since for the function (5) we have

$$\frac{d^2y}{dx^2} = -c_1 \sin x,$$

and hence

$$\frac{d^2y}{dx^2} + y = -c_1 \sin x + c_1 \sin x \equiv 0,$$

as required by (6).

Similarly, the function

$$y = c_2 \cos x, \tag{7}$$

where c_2 is again an arbitrary constant, is also a solution of (6) since we may then write

$$\frac{d^2y}{dx^2} = - c_2 \cos x,$$

and hence

$$\frac{d^2y}{dx^2} + y = - c_2 \cos x + c_2 \cos x \equiv 0.$$

The functions (5) and (7) illustrate what are known as **particular solutions** of a differential equation; namely, of the equation (6). If these two particular solutions be added together, thus obtaining the function

$$y = c_1 \sin x + c_2 \cos x, \tag{8}$$

we again have a solution of (6), as is easily verified. Moreover, it can be shown (though the proof lies beyond the present text) that every particular solution of (6) can be obtained from (8) by assigning suitable values to the constants c_1, c_2 which there appear. For example, the particular solution (5) results by placing in (8) $c_1 = c_1$, $c_2 = 0$, while the particular solution (7) results by placing $c_1 = 0$, $c_2 = c_2$. The function (8) thus illustrates what is known as the **general solution** of a differential equation. Analogously, the general solution of any differential equation is a solution which may be made to give any particular solution whatever by suitably assigning values to the arbitrary constants which it contains.

An **ordinary** differential equation is one in which only one independent variable enters, as in (1), (2), (3). A **partial** differential equation is one in which more than one independent variable enters, as in (4).

The **order** of a differential equation is that of the highest derivative, or differential, appearing in it. Thus, (2), (3), (4) are of the first order while (1) is of the second order.

Thus, the differential equation

$$x\frac{dy}{dx} + y = 0$$

would be described as an ordinary differential equation of the first order.

Similarly, the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

is a partial differential equation of the second order.

EXERCISES

In the following list the function y defined, either explicitly or implicitly, in the second column is a solution of the corresponding differential equation on the left, c, c_1, c_2, etc. being arbitrary constants. Verify this fact.

Differential equation	*Solution*
1. $\dfrac{d^2y}{dx^2} - y = 0.$	$y = c_1e^x + c_2e^{-x}.$
2. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0.$	$y = (c_1 + c_2x)e^{2x}.$
3. $\dfrac{d^2y}{dx^2} + y = \cos ax.$	$y = c_1 \sin x + c_2 \cos x + \dfrac{\cos ax}{1 - a^2}.$
4. $(1 + y)\,dx - (1 - x)\,dy = 0.$	$(1 + y)(1 - x) = c.$
5. $\sqrt{1 - x^2}\,dy + \sqrt{1 - y^2}\,dx = 0.$	$y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = c.$
6. $(x^2 + y^2)\,dx = 2\,xy\,dy.$	$y^2 = x^2 + cx.$
7. $3\,y^2\dfrac{dy}{dx} - ay^3 = x + 1.$	$y^3 = ce^{ax} - \dfrac{x + 1}{a} - \dfrac{1}{a^2}.$
8. $dy - \dfrac{xy}{1 + x^2}\,dx = \dfrac{a}{1 + x^2}\,dx.$	$y = ax + c(1 + x^2)^{\frac{1}{2}}.$
9. $(x^2y + x)\,dy + (xy^2 - y)\,dx = 0.$	$xy + \log\dfrac{y}{x} = c.$
10. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 0.$	$y = c_1e^{3x} + c_2e^{-2x} + c_3.$

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Even though a differential equation be of only the first order, it is not always possible to arrive at a solution of it by elementary methods. However, certain special types of such equations may be readily solved as we shall now show. We shall confine the discussion to a few types that are especially important in their applications.

163. Type I. Variables separable. If a differential equation can be arranged in the form

$$f(x)\,dx + F(y)\,dy = 0, \tag{1}$$

where $f(x)$ is a function of x alone and $F(y)$ is a function of y alone, the variables of the equation are said to be **separable.** The general solution of such an equation is always obtainable by direct integration, as illustrated in the following examples.

Example 1. Solve the differential equation

$$(1 - y)\,dx - (1 + x)\,dy = 0. \tag{2}$$

SOLUTION. By dividing both members of the equation by $(1 - y)(1 + x)$ it takes the form

$$\frac{dx}{1 + x} - \frac{dy}{1 - y} = 0, \tag{3}$$

which is seen to be of the type (1), the variables being now separated.

Integrating (3) we obtain

$$\int \frac{dx}{1 + x} - \int \frac{dy}{1 - y} = c,$$

or

$$\log(1 + x) + \log(1 - y) = c.$$

Hence

$$\log(1 + x)(1 - y) = c,$$

or

$$(1 + x)(1 - y) = e^c. \tag{4}$$

Here, inasmuch as c is an arbitrary constant, the expression e^c is likewise an arbitrary constant. Hence (4) may be replaced by the simpler form

$$(1 + x)(1 - y) = c. \tag{5}$$

The function y thus defined is, therefore, a solution of (2). Moreover, it is the *general* solution inasmuch as the foregoing reasoning shows that *any* function y which satisfies (2) must necessarily satisfy (5).

Example 2. Solve the differential equation

$$y(x + 1)\,dx - x(y - 1)\,dy = 0. \tag{6}$$

SOLUTION. By dividing through by xy the variables become separated, the equation taking the form

$$\frac{x + 1}{x}\,dx - \frac{y - 1}{y}\,dy = 0.$$

Hence

$$\int \frac{x + 1}{x}\,dx - \int \frac{y - 1}{y}\,dy = c,$$

or

$$x + \log x - y + \log y = c,$$

or

$$\log xy + x - y = c.$$

The function y thus determined is, for the reasons indicated in connection with the solution of Ex. 1, the general solution of (6), c being an arbitrary constant.

EXERCISES

Solve each of the following differential equations:

1. $y\,dx - x\,dy = 0.$ *Ans.* $y = cx.$
2. $x\,dy + 2\,y\,dx = 0.$ *Ans.* $x^2y = c.$
3. $x^2\,dy + y^2\,dx = 0.$ *Ans.* $x + y = cxy.$
4. $\cot y\,dx + \cot x\,dy = 0.$ *Ans.* $\cos y \cos x = c.$
5. $x^2y\,dx + (1 - x)(1 - y^2)\,dy = 0.$ *Ans.* $\log \frac{y}{1 - x} = x + \frac{1}{2}(x^2 + y^2) + c.$
6. $x^2(1 - y)\,dy + y^2(1 + x)\,dx = 0.$ *Ans.* $\frac{1}{x} + \frac{1}{y} + \log \frac{y}{x} = c.$
7. $x(1 - y^2)\,dx + y(1 - x^2)\,dy = 0.$ *Ans.* $x^2 + y^2 = x^2y^2 + c.$
8. $\sqrt{1 - y^2}\,dx + \sqrt{1 - x^2}\,dy = 0.$ *Ans.* $y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = c.$
9. $\frac{y}{1 + x} + 2\,x \log y \frac{dy}{dx} = 0.$ *Ans.* $(\log y)^2 = \log \frac{c(1 + x)}{x}.$
10. $(y^2 - 1)\,dx + y(x^2 + 1)\,dy = 0.$ *Ans.* $\arctan x + \frac{1}{2} \log (y^2 - 1) = c.$
11. $y\,dx + x(1 - y)\,dy = 0.$ *Ans.* $xy = ce^y.$
12. $(1 + x^2)\,dy = (1 + y^2)\,dx.$
13. $xy\,dx - (a + x)(b + y)\,dy = 0.$
14. $x^2\,dx - y\,dy + x^2y^2\,dx + x^3y\,dy = 0.$
15. $\sec^2 x \tan y\,dy + \sec^2 y \tan x\,dx = 0.$
16. $\sin y\,dy + \sin x \cos y\,dx = 0.$
17. $\cos y\,dx + x \sin y\,dy = 0.$
18. $x \csc y\,dx + y^2 \csc x\,dy = 0.$
19. $e^{x+2y}\,dx + y \sec x\,dy = 0.$
20. $x^2y^2\,dx - e^x \log y\,dy = 0.$
21. $\sin^2 y\,dy - \sin^2 x \cos y\,dx = 0.$

164. Type II. Equations with homogeneous coefficients. A function $M(x, y)$ is said to be a **homogeneous function** *of degree n* in the two variables x, y if, when x and y are replaced respectively by kx and ky, the result is the original function multiplied by k^n.

Thus, the function
$$M(x, y) = x^2 + xy$$
is a homogeneous function of the second degree in x and y since the result of placing therein $x = kx$, $y = ky$ is
$$(kx)^2 + (kx)(ky) = k^2(x^2 + xy) = k^2M(x, y).$$

It is now to be remarked that any differential equation of the form
$$M(x, y)\,dx + N(x, y)\,dy = 0, \qquad (1)$$
wherein M and N are any two homogeneous functions of the *same* degree, may be solved by making the substitution
$$y = vx.$$
Such a substitution, in fact, converts (1) into a new equation in which the variables x and v are separable and which therefore may be solved as indicated in Art. 163. This having been done, the solution of (1) itself is obtainable.

Example. Solve the differential equation

$$y^2\,dx + (x^2 - xy)\,dy = 0. \tag{2}$$

SOLUTION. Here

$$M(x, y) = y^2 \text{ and } N(x, y) = x^2 - xy.$$

These are homogeneous functions of the same degree, namely, the second, in x and y. Hence we place

$$y = vx \tag{3}$$

from which we have also

$$dy = v\,dx + x\,dv. \tag{4}$$

Substituting (3) and (4) in (2), the latter takes the form

$$v^2x^2\,dx + (x^2 - vx^2)(v\,dx + x\,dv) = 0,$$

or

$$x^2v\,dx + x^3(1 - v)\,dv = 0,$$

or

$$v\,dx + x(1 - v)\,dv = 0.$$

To separate the variables, divide by vx, obtaining

$$\frac{dx}{x} + \left(\frac{1}{v} - 1\right)dv = 0.$$

Integrating,

$$\log x + \log v - v + c = 0. \tag{5}$$

Let us take c for convenience in the form $\log c$, in which case (5) takes the form

$$\log x + \log v + \log c = v,$$

or

$$\log cvx = v. \tag{6}$$

Now replace v by its equivalent $\frac{y}{x}$, as obtained from (3). Then (6) takes the form

$$\log cy = \frac{y}{x}, \tag{7}$$

which may be regarded as the general solution of the given equation (2), inasmuch as any function y satisfying (2) has thus been shown to be determined by (7).

We observe finally that (7) may be put in the form

$$cy = e^{\frac{y}{x}}.$$

The student should note that in this as in all other similar examples the auxiliary variable v does not appear in the final solution.

EXERCISES

Solve each of the following differential equations:

1. $y\,dx + (y - x)\,dy = 0.$ *Ans.* $cy = e^{-\frac{x}{y}}.$
2. $(x + y)\,dx + x\,dy = 0.$ *Ans.* $x^2 + 2\,xy = c.$
3. $(x^2 + y^2)\,dx - 2\,xy\,dy = 0.$ *Ans.* $y^2 = x^2 + cx.$
4. $xy^2\,dy = (x^3 + y^3)\,dx.$ *Ans.* $y^3 = 3\,x^3 \log cx.$
5. $(x^2 + y^2)\,dy + xy\,dx = 0.$ *Ans.* $y^2(2\,x^2 + y^2) = c.$
6. $(y + \sqrt{x^2 - y^2})\,dx - x\,dy = 0.$ *Ans.* $\log cx = \operatorname{arc\,sin} \frac{y}{x}.$
7. $xy\,dx - (x^2 + xy + y^2)\,dy = 0.$ *Ans.* $x + y = ce^{\frac{x}{y}}.$
8. $x\,dx + (y - x)\,dy = 0.$ *Ans.* $\log c(y^2 - xy + x^2) = \frac{2}{\sqrt{3}} \operatorname{arc\,tan} \frac{2\,y - x}{\sqrt{3}\,x}.$
9. $x\frac{dy}{dx} + 3\,y = 4\,x.$ *Ans.* $x^3(y - x) = c.$
10. $\frac{dy}{dx} + \frac{x}{y} = 2.$ *Ans.* $(x - y)e^{\frac{x}{x-y}} = c.$
11. $(2\sqrt{xy} - y)\,dx + x\,dy = 0.$
12. $(x^2 - xy + y^2)\,dx + x^2\,dy = 0.$
13. $(x + y)\,dx = x\,dy.$
14. $(x^2 + y^2)\,dx = xy\,dy.$
15. $(x^2 + y^2 + xy)\,dx = x^2\,dy.$
16. $y(x^2 - y^2)\,dx = x^3\,dy.$
17. $(xy - x^2 - y^2)\,dx = x^2\,dy.$
18. $(xy - y\sqrt{y^2 - x^2})\,dx = x^2\,dy.$
19. $(xy - x^2)\cos\frac{x}{y}\,dy = y^2\,dx.$
20. $(xy^2 e^{\frac{y^2}{x^2}} + x^3)\,dx = x^2\,ye^{\frac{y^2}{x^2}}\,dy.$

165. Type III. Linear equations. A differential equation of the first order is called **linear** in case it has the type-form

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only, or constants. The fact that in such an equation the dependent variable y and its derivative $\frac{dy}{dx}$ occur to the *first* power only accounts for the name *linear* as applied to such equations, the analogy between them and the equations of straight lines in analytic geometry being thus apparent. The general solution of every equation of the type (1) may be found, as indicated in the following theorem:

Theorem. *The general solution of the linear differential equation*

$$\frac{dy}{dx} + Py = Q$$

is given by the formula

$$y = e^{-\int P\,dx}\left(\int Qe^{\int P\,dx}dx + c\right),$$

where c is an arbitrary constant.

PROOF. In (1) place

$$y = uz, \tag{2}$$

where both u and z are functions of x as yet to be determined. Then

$$\frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx}. \tag{3}$$

Substituting (2) and (3) in (1), it takes the form

$$u\frac{dz}{dx} + z\frac{du}{dx} + Puz = Q$$

or

$$u\frac{dz}{dx} + \left(\frac{du}{dx} + Pu\right)z = Q. \tag{4}$$

Let us now determine the hitherto unknown function u in such a manner that the coefficient of z here appearing shall vanish; that is, such that

$$\frac{du}{dx} + Pu = 0.$$

This equation gives, upon separating the variables,

$$\frac{du}{u} = -P\,dx.$$

It follows upon integrating that

$$\log u = -\int P\,dx + c_1$$

or

$$u = e^{c_1}e^{-\int P\,dx} = k_1e^{-\int P\,dx}, \tag{5}$$

where k_1 is an arbitrary constant.

With u thus determined, (4) becomes

$$k_1 e^{-\int P\,dx} \frac{dz}{dx} = Q, \tag{6}$$

from which z may now be determined as follows:

We have from (6)

$$dz = \frac{1}{k_1} Q e^{\int P\,dx}\, dx,$$

hence

$$z = \frac{1}{k_1} \int Q e^{\int P\,dx}\, dx + k_2, \tag{7}$$

where k_2 is another arbitrary constant.

Substituting in (2) the expressions (5) and (7) for u and z, the desired expression for y becomes

$$y = e^{-\int P\,dx}\left(\int Q e^{\int P\,dx}\, dx + c\right),$$

where c is an arbitrary constant.

The proof is thus complete.

Example. Solve the differential equation

$$\frac{dy}{dx} + \frac{2\,xy}{1 - x^2} = 1 - x^2.$$

SOLUTION. We here have

$$P = \frac{2\,x}{1 - x^2}, \qquad Q = 1 - x^2.$$

Hence,

$$\int P\,dx = \int \frac{2\,x\,dx}{1 - x^2} = -\log(1 - x^2),$$

and therefore

$$e^{\int P\,dx} = e^{-\log(1 - x^2)} = \frac{1}{1 - x^2}. \tag{8}$$

It follows that

$$\int Q e^{\int P\,dx}\, dx = \int (1 - x^2)\frac{dx}{1 - x^2} = \int dx = x.$$

Observing that (8) gives also

$$e^{-\int P\,dx} = e^{\log(1 - x^2)} = 1 - x^2,$$

it follows by application of the theorem that the desired solution y is given by the equation

$$y = (1 - x^2)(x + c).$$

EXERCISES

Solve each of the following differential equations:

1. $\dfrac{dy}{dx} + y = x.$ *Ans.* $y = ce^{-x} + x - 1.$

2. $\dfrac{dy}{dx} + \dfrac{x}{1 + x^2} y = 3x.$ *Ans.* $y = 1 + x^2 + c(1 + x^2)^{-\frac{1}{2}}.$

3. $\dfrac{dy}{dx} - \dfrac{2y}{x + 1} = (x + 1)^3.$ *Ans.* $2y = (x + 1)^2[(x + 1)^2 + c].$

4. $x\dfrac{dy}{dx} + y = (1 - x^2)^{-\frac{1}{2}}.$ *Ans.* $y = \dfrac{1}{x}(\text{arc} \sin x + c).$

5. $x\,dy - 3y\,dx = x^2\,dx.$ *Ans.* $y = cx^3 - x^2.$

6. $x\,dy + 3y\,dx = 4x\,dx.$ *Ans.* $y = \dfrac{c}{x^3} + x.$

7. $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{x + 1}{x}.$ *Ans.* $y = \dfrac{x}{3} + \dfrac{1}{2} + \dfrac{c}{x^2}.$

8. $\dfrac{ds}{dt} + \dfrac{2st}{1 - t^2} = (1 - t^2)^{\frac{1}{2}}.$ *Ans.* $s = (1 - t^2)(\text{arc} \sin t + c).$

9. $\dfrac{dy}{dx} - \dfrac{xy}{1 + x^2} = \dfrac{a}{1 + x^2}.$ *Ans.* $y = ax + c(1 + x^2)^{\frac{1}{2}}.$

10. $\dfrac{dy}{dx} + y \tan x = a \tan x.$ *Ans.* $y = a + c \cos x.$

11. $(1 - x^2)^{\frac{1}{2}} \dfrac{dy}{dx} + y = 1.$ *Ans.* $y = 1 + ce^{-\text{arc} \sin x}.$

12. $\dfrac{dy}{dx} + y \cos x = \sin 2x.$

13. $\dfrac{dy}{dx} + \dfrac{1 - 2x}{x^2} y = 1.$

14. $\dfrac{dy}{dx} + y \sin x = \dfrac{1}{2} \sin 2x.$

15. $\dfrac{dy}{dx} + 2xy = e^{-x^2} \cos x.$

16. $\dfrac{du}{dv} = 3u + v.$

17. $\dfrac{dv}{du} = 3u + v.$

18. $x\dfrac{dy}{dx} + y = x + 1.$

19. $x^2\dfrac{dy}{dx} + y = 1.$

20. $x\dfrac{dy}{dx} - 2y = x^3 \cos x.$

21. $x\dfrac{dy}{dx} - y = x^2 \sec^2 x.$

22. $(x + 1)\dfrac{dy}{dx} - xy = 0.$

23. $x\dfrac{dy}{dx} - 3y = x^3.$

DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

In considering differential equations of the second order we shall confine ourselves to three types which are particularly important in their applications. The subject as a whole is very extensive, thus lying beyond the scope of the present text.

166. Type I. Homogeneous linear equation with constant coefficients. This is the type

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{1}$$

where p and q are constants.

We proceed to prove three theorems concerning the solution of (1) as follows:

Theorem I. *In case the two roots m_1, m_2 of the quadratic equation* (auxiliary equation)

$$m^2 + pm + q = 0 \tag{2}$$

are real and distinct, the general solution of (1) *may be expressed in the form*

$$y = c_1e^{m_1x} + c_2e^{m_2x}, \tag{3}$$

where c_1 and c_2 are arbitrary constants.

PROOF. The substitution of e^{mx} for y in (1), m being a constant yet to be determined, gives the equation the form

$$e^{mx}(m^2 + pm + q) = 0.$$

This equation will evidently be satisfied if to m we assign either the value m_1 or m_2, where these are the two roots of (2). It follows that e^{m_1x} and e^{m_2x} are particular solutions of (1). If we now multiply each of these expressions by arbitrary constants and add the results, thus obtaining the function (3), we arrive at an expression which likewise satisfies (1), as appears by direct substitution of (3) in (1). Hence (3) is a solution of (1). Proof that it is the *general* solution lies beyond the present text.

Example. Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

SOLUTION. The auxiliary equation is

$$m^2 - 3m + 2 = 0,$$

the roots of which are $m_1 = 1$ and $m_2 = 2$.

Hence, applying the theorem, the general solution is

$$y = c_1e^x + c_2e^{2x}.$$

Theorem II. *In case the two roots m_1, m_2 of the auxiliary equation* (2) *are conjugate imaginary quantities, so that*

$$m_1 = a + ib, \qquad m_2 = a - ib, \tag{4}$$

where a and b are real numbers and $i = \sqrt{-1}$, the general solution of (1) *may be expressed in the form*

$$\boldsymbol{y = e^{ax}(c_1 \cos bx + c_2 \sin bx),}$$

where c_1 and c_2 are arbitrary constants.

PROOF. Consider the function

$$y_1 = c_1e^{ax}\cos bx, \tag{5}$$

where a and b are real quantities such that $a + ib$ is a root of (2).

Then we shall have

$$y_1' = c_1e^{ax}(-b\sin bx + a\cos bx),$$

and hence we find

$$y_1'' = c_1e^{ax}(a^2\cos bx - b^2\cos bx - 2ab\sin bx).$$

Using the foregoing expressions for y_1, y_1', and y_1'', the left member of (1) takes the following form:

$$y_1'' + py_1' + qy_1$$
$$= c_1e^{ax}[a^2 - b^2 + pa + q]\cos bx - c_1e^{ax}[2ab + pb]\sin bx. \tag{6}$$

Now, $a + ib$ being by hypothesis a root of the quadratic equation (2), we have

$$(a + ib)^2 + p(a + ib) + q = 0,$$

which, upon expanding $(a + ib)^2$ and recalling that $i^2 = -1$, becomes

$$(a^2 - b^2 + pa + q) + i(2ab + pb) = 0.$$

This equation is possible, as shown in elementary algebra, only in case the real part and the pure imaginary part are separately equal to zero; that is, it requires that

$$a^2 - b^2 + pa + q = 0, \qquad 2\,ab + pb = 0. \tag{7}$$

But equations (7) cause the right member of (6), and hence the left member also, to be equal to zero for all values of x; that is, we have

$$y_1'' + py_1' + qy_1 = 0. \tag{8}$$

The function y_1 is therefore a particular solution of (1).

Likewise, the function

$$y_2 = c_2 e^{ax} \sin bx$$

is found to be such that

$$y_2'' + py_2' + qy_2 \equiv 0. \tag{9}$$

Hence y_2 is also a particular solution of (1).

Consider finally the function

$$y = y_1 + y_2 = e^{ax}(c_1 \cos bx + c_2 \sin bx). \tag{10}$$

For this function we have

$$y'' + py' + qy = (y_1'' + py_1' + q) + (y_2'' + py_2' + q),$$

and the second member is here equal to zero by virtue of (8) and (9). Hence (10) is a solution of (1), as stated in the theorem. Proof that it is the general solution lies beyond the present text.

Example. Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13\,y = 0.$$

SOLUTION. The auxiliary equation is

$$m^2 - 4\,m + 13 = 0,$$

the roots of which are

$$m_1 = 2 + 3\,i, \qquad m_2 = 2 - 3\,i.$$

We here have, therefore,

$$a = 2, \qquad b = 3.$$

Hence, applying the theorem, the general solution is

$$y = e^{2x}\,(c_1 \cos 3\,x + c_2 \sin 3\,x).$$

Theorem III. *In case the two roots m_1, m_2 of the auxiliary equation are equal to each other, each being equal to m, the general solution of* (1) *may be expressed in the form*

$$y = c_1e^{mx} + c_2xe^{mx},$$

where c_1 and c_2 are arbitrary constants.

PROOF. The function

$$y_1 = c_1e^{mx}$$

is at once found to satisfy (1). See the proof of Theorem I.

Consider now the function

$$y_2 = c_2xe^{mx}.$$

For this we find

$$y_2' = c_2e^{mx}(1 + mx), \qquad y_2'' = c_2e^{mx}(2\,m + m^2x).$$

Substituting these for y, y', and y'' respectively in (1), the equation takes the following form, after collection of terms:

$$c_2e^{mx}\{(m^2 + pm + q)x + (2\,m + p)\} = 0. \tag{11}$$

In order to show that y_2 is a solution of (1) it thus remains but to show that (11) is satisfied when m is, as we have supposed, one of the two *equal* roots of the auxiliary equation

$$m^2 + pm + q = 0. \tag{12}$$

Now, inasmuch as this quadratic equation has equal roots, we know from elementary algebra that we must have $p^2 = 4\,q$, or $q = \frac{1}{4}\,p^2$. Thus (12) may be written

$$m^2 + pm + \tfrac{1}{4}\,p^2 = 0,$$

from which we obtain $m = -\dfrac{p}{2}$, or

$$2\,m + p = 0. \tag{13}$$

Equations (12) and (13) show that (11) is satisfied, as desired, for all values of x and hence that y_2 is a solution of (1).

It follows that the function

$$y = y_1 + y_2 = c_1e^{mx} + c_2xe^{mx} \tag{14}$$

is a solution of (1) as stated in the theorem, since we may now write

$$y'' + py' + q = (y_1'' + py_1' + qy_1) + (y_2'' + py_2' + qy_2) \equiv 0 + 0 \equiv 0.$$

Proof that (14) is the general solution lies beyond the present text and must therefore be omitted.

Example. Solve the differential equation $y'' - 4y' + 4y = 0$.

SOLUTION. The auxiliary equation is

$$m^2 - 4m + 4 = (m - 2)^2 = 0,$$

the roots of which are each equal to 2. Applying the theorem, the general solution is therefore

$$y = c_1e^{2x} + c_2xe^{2x} \quad \text{or} \quad y = e^{2x}(c_1 + c_2x).$$

EXERCISES

Solve each of the following differential equations:

1. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 4y = 0.$ *Ans.* $y = c_1e^{4x} + c_2e^x$.
2. $4y'' + 8y' - 5y = 0.$ *Ans.* $y = c_1e^{\frac{1}{2}x} + c_2e^{-\frac{5}{2}x}$.
3. $y'' - 4y' + y = 0.$ *Ans.* $y = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$.
4. $y'' - 6y' + 10y = 0.$ *Ans.* $y = e^{3x}(c_1 \cos x + c_2 \sin x)$.
5. $y'' + 9y = 0.$ *Ans.* $y = c_1 \cos 3x + c_2 \sin 3x$.
6. $2y'' + 2y' + 3y = 0.$ *Ans.* $y = e^{-\frac{1}{2}x}\left(c_1 \cos \dfrac{\sqrt{5}}{2}x + c_2 \sin \dfrac{\sqrt{5}}{2}x\right)$.
7. $y'' + 6y' + 9y = 0.$ *Ans.* $y = e^{-3x}(c_1 + c_2x)$.
8. $4y'' - 4y' + y = 0.$ *Ans.* $y = e^{\frac{1}{2}x}(c_1 + c_2x)$.
9. $q^2y'' - 2pqy' + p^2y = 0.$ *Ans.* $y = e^{\frac{p}{q}x}(c_1 + c_2x)$.
10. $9y'' + 6y' - 35y = 0.$
11. $y'' + y' + y = 0.$
12. $9y'' - 12y' + 4y = 0.$
13. $4y'' - y' - 3y = 0.$
14. $y'' + 4y' + 12y = 0.$
15. $3y'' - 2y' - 5y = 0.$
16. $3y'' - 7y' + 2y = 0.$
17. $36y'' + 60y' + 25y = 0.$
18. $2y'' + 3y' - y = 0.$
19. $3y'' + 2y' - 4y = 0.$

167. Equations of the type

$$\frac{d^2y}{dx^2} = X,$$

where X is a function of x only, or a constant.

The general solution of any equation of this type may be obtained by the process illustrated in the following example.

Example. Solve the differential equation $y'' = x \sin x$.

SOLUTION. We have at once

$$y' = \int x \sin x \, dx + c_1 = - x \cos x + \sin x + c_1.$$

Integrating again,

$$\begin{aligned} y &= -\int x \cos x + \int \sin x + \int c_1 \, dx + c_2 \\ &= (- x \sin x - \cos x) - \cos x + c_1x + c_2 \\ &= - x \sin x - 2 \cos x + c_1x + c_2. \end{aligned}$$

168. Equations of the type

$$\frac{d^2y}{dx^2} = Y,$$

where Y is a function of y only, or a constant.

The general solution of an equation of this type may be obtained by the process illustrated in the following example.

Example. Solve the differential equation

$$\frac{d^2y}{dx^2} = \frac{a}{y^3}.$$

SOLUTION. First, multiply the left member by $2\frac{dy}{dx}\,dx$ and the right member by the equivalent expression $2\,dy$. We thus obtain

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}\,dx = \frac{2\,a}{y^3}\,dy,$$

which may be written

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 dx = \frac{2\,a}{y^3}\,dy.$$

Integrating,

$$\left(\frac{dy}{dx}\right)^2 = 2\,a\int\frac{dy}{y^3} = -\frac{a}{y^2} + c_1 = \frac{c_1y^2 - a}{y^2}.$$

Hence, extracting the square root of each member,

$$\frac{dy}{dx} = \pm\frac{1}{y}\sqrt{c_1y^2 - a}.$$

Separating the variables, this equation becomes

$$\pm\frac{y\,dy}{\sqrt{c_1y^2 - a}} = dx.$$

Integrating again, we obtain

$$\pm\frac{1}{c_1}\sqrt{c_1y^2 - a} = x + c_2 \qquad \text{or} \qquad \pm\sqrt{c_1y^2 - a} = c_1x + c_1c_2.$$

In view of the arbitrariness of c_1 and c_2, this equation is equivalent to

$$\pm\sqrt{c_1y^2 - a} = c_1x + c_2,$$

from which we obtain

$$c_1y^2 - a = (c_1x + c_2)^2.$$

The desired solution y is therefore determined by the equation

$$c_1y^2 = (c_1x + c_2)^2 + a.$$

EXERCISES

1. Solve each of the following differential equations.

(*a*) $\frac{d^2y}{dx^2} = x^2.$ *Ans.* $y = \frac{1}{12}x^4 + c_1x + c_2.$

(*b*) $\frac{d^2s}{dt^2} = \log t.$ *Ans.* $s = \frac{1}{2}t^2 \log t - \frac{3}{4}t^2 + c_1t + c_2.$

(*c*) $\frac{d^2y}{dx^2} = \sin x.$ *Ans.* $y = -\sin x + c_1x + c_2.$

(*d*) $\frac{d^2y}{dx^2} = x^2 \sin x.$ *Ans.* $y = (6 - x^2)\sin x - 4x\cos x + c_1x + c_2.$

(*e*) $y'' = \frac{1}{\sqrt{1 - x^2}}.$ *Ans.* $y = x \arcsin x + \sqrt{1 - x^2} + c_1x + c_2.$

(*f*) $y'' = \sin^2 x.$ *Ans.* $y = \frac{1}{4}x^2 + \frac{1}{8}\cos 2x + c_1x + c_2.$

2. If y be such that

$$\frac{d^2y}{dx^2} = y^3,$$

find y, it being given that $\frac{dy}{dx} = \sqrt{2}$ when $y = \sqrt{2}.$ *Ans.* $y(x + c) + \sqrt{2} = 0.$

HINT. Proceeding as in the Example of Art. 168, we first find that

$$\left(\frac{dy}{dx}\right)^2 = 2\int y^3\,dy = \frac{1}{2}y^4 + c_1.$$

But since $\frac{dy}{dx} = \sqrt{2}$ when $y = \sqrt{2}$, this equation determines c_1; namely, we must have $c_1 = 0$. Continue now as in Art. 168.

3. Determine s if

$$\frac{d^2s}{dt^2} = 2 \sin 2s$$

and it is given that $\frac{ds}{dt} = \sqrt{2}$ when $s = \frac{\pi}{4}$. *Ans.* $-2t + \log c(\csc s - \cot s) = 0.$

4. Determine y if

$$\frac{d^2y}{dx^2} = \sec^2 y \tan y,$$

it being given that $\frac{dy}{dx} = 1$ when $y = \frac{\pi}{4}.$

169. Summary. It is to be understood that the foregoing discussion treats but a few of the types of differential equations of the first and second orders for which general solutions are readily obtainable. The types selected have immediate applications in physics, as will be illustrated in the remaining articles of this chapter. The study of differential equations in general, including equations of higher orders than the second, constitutes an extensive special branch of mathematics

APPLICATIONS OF DIFFERENTIAL EQUATIONS

170. The pendulum. In Fig. 171 let P be a particle connected to the fixed point O by a weightless cord or rod of length l and suppose that the particle is allowed to oscillate along the circular arc MP under the action of gravity, thus having the motion of a pendulum. If the motion starts from rest at some fixed point as P_0, the problem is to determine both the position and the velocity of the particle at any specified time t.

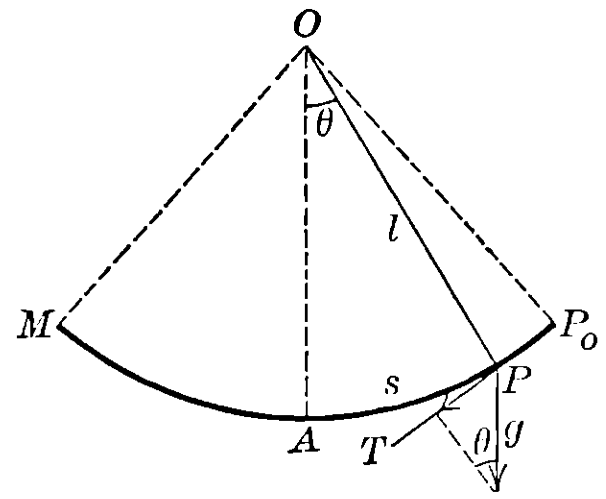


FIG. 171

Let OA be the vertical and denote by θ the angle AOP as measured from OA; also let s be the length of the arc AP as measured from A. Then the acceleration at P is equal by Art. 55 to $-\frac{d^2s}{dt^2}$, the minus sign being due to the fact that the velocity is always decreasing in the direction AP along which s is measured. But the acceleration at P is likewise equal by elementary physics to the component in the tangential direction PT of the acceleration g acquired by a freely falling body, this component being in the present instance $g \sin \theta$. We therefore arrive at the following differential equation characterizing the motion at any time t:

$$-\frac{d^2s}{dt^2} = g \sin \theta.$$

But $s = l\theta$, so that, after changing the sign of each member, we obtain

$$l\frac{d^2\theta}{dt^2} = -g \sin \theta. \tag{1}$$

This equation is of the type considered in Art. 168, hence we proceed to solve it in the manner there indicated.

Multiplying both members by $2\,d\theta$, the equation takes the form

$$l\frac{d}{dt}\left(\frac{d\theta}{dt}\right)^2 dt = -2g \sin \theta \, d\theta.$$

Integrating, we obtain

$$l\left(\frac{d\theta}{dt}\right)^2 = -2g\int \sin \theta \, d\theta = 2g \cos \theta + c, \tag{2}$$

where the value of the constant c is to be determined from the fact that at the starting point P_0, or at $\theta = \theta_0$, the value of the angular velocity,

or $\frac{d\theta}{dt}$, is zero. This condition when introduced in (2) gives at once $c = -2g\cos\theta_0$, so that the final form of (2) is

$$l\left(\frac{d\theta}{dt}\right)^2 = 2g(\cos\theta - \cos\theta_0). \tag{3}$$

Multiplying both members by l and then taking square roots, noting that $l\frac{d\theta}{dt}$ is equal to the velocity v of the particle, we arrive in summary at the following result:

The velocity v of the pendulum of length l at any position-angle θ is determined by the formula

$$v = \sqrt{2gl(\cos\theta - \cos\theta_0)},$$

where θ_0 is the initial value of θ at which the pendulum starts from rest.

Thus one part of the problem proposed at the beginning becomes completely answered. In order to answer the other part, namely, to determine the position-angle θ corresponding to any given time t, we return to (3), which, after taking the square root of each member and then separating the variables, may be written in the form

$$\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{\frac{2g}{l}}\,dt. \tag{4}$$

Naturally, the desired relation connecting θ with t will result by integrating this equation member by member and subsequently determining the value of the constant of integration from the fact that $\theta = \theta_0$, when $t = 0$. However, the left member of (4) is not an integrable form; that is, it cannot be evaluated in terms of the so-called elementary functions such as have thus far been considered in this text. We are therefore forced to abandon the undertaking, but we shall now show that for the case of most practical importance, namely, that in which θ_0 and hence θ are confined to very small values, it is easy to obtain an equation for θ which gives a very good approximation to its correct value corresponding to any given time t.

With θ thus limited to very small values, we may substitute θ for $\sin\theta$ in (1) with but slight error, in which case the equation may be written

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

We here meet with a differential equation of the type II, Art. 166, the general solution of which by the theorem of that article is readily found to be

$$\theta = c_1 \cos\sqrt{\frac{g}{l}}\, t + c_2 \sin\sqrt{\frac{g}{l}}\, t, \tag{5}$$

where c_1 and c_2 are to be determined from the initial conditions; namely, $\theta = \theta_0$ when $t = 0$ and $\frac{d\theta}{dt} = 0$ when $t = 0$. Thus we have $c_1 = \theta_0$, $c_2 = 0$. Hence we arrive in summary at the following result:

If the oscillations of the pendulum of length l are confined to a small range about the vertical, the position-angle θ at the time t is determined approximately by the formula

$$\boldsymbol{\theta = \theta_0 \cos\sqrt{\frac{g}{l}}\, t,}$$

where θ_0 is the initial value of θ at which the pendulum starts from rest.

In applying the foregoing results it is to be remembered that if t is measured in seconds and l in feet, then $g = 32$ (approximately). Both θ_0 and θ are, of course, measured in radians.

EXERCISES

1. If a pendulum starts from the position $\theta_0 = 90°$ (Fig. 171), show that it passes through the vertical with a velocity equal to that of a body falling freely through a height equal to the length of the pendulum.

2. If a pendulum of length 1 foot starts from the position $\theta_0 = 60°$, find the velocity with which it passes through the vertical.
Ans. $\sqrt{32}$ ft. per sec., approximately.

3. If a pendulum starts at the position $\theta_0 = 60°$, find at what position it will have acquired half its maximum velocity. *Ans.* $\theta = 51°\,19'$

4. Show that the time t_1 of one swing or beat of a pendulum of length l and of small oscillation is given approximately by the formula

$$t_1 = \pi\sqrt{\frac{l}{g}}.$$

5. Show that the length of a pendulum of small oscillation that makes one beat per second is $\frac{g}{\pi^2} = 3.26$ ft., approximately.

6. Find the length of the pendulum of small oscillation which makes 20 beats per minute. *Ans.* 29.33 ft.

7. Show that no pendulum such as shown in Fig. 171 can be constructed which will attain half its maximum velocity when half way to its lowest point.

8. A small heavy bead is arranged to slide on a smooth circular hoop of radius l lying in a vertical plane. If the bead be started in a downward direction from the point P_0 whose central angle is θ_0 (Fig. 171), find the least initial velocity v_0 which it must have in order to make a complete revolution about the hoop, friction being neglected.

171. Attraction of gravitation. It is shown in the theory of gravitational attraction that the acceleration of a body falling from a great distance toward the earth varies inversely as the square of the distance of the body from the earth's center. Thus, if in Fig. 172 the body starts from rest at P_0, its acceleration α at any point such as P satisfies an equation of the form

$$\alpha = \frac{k}{\overline{OP}^2}, \tag{1}$$

where k is a constant. The value of k is, moreover, readily determined, for when the body reaches the surface of the earth, that is, when $OP = R$, the radius of the earth, we know that $\alpha = g$, where g is the familiar constant met with, for example, in the last article. Introducing this fact in (1), we obtain

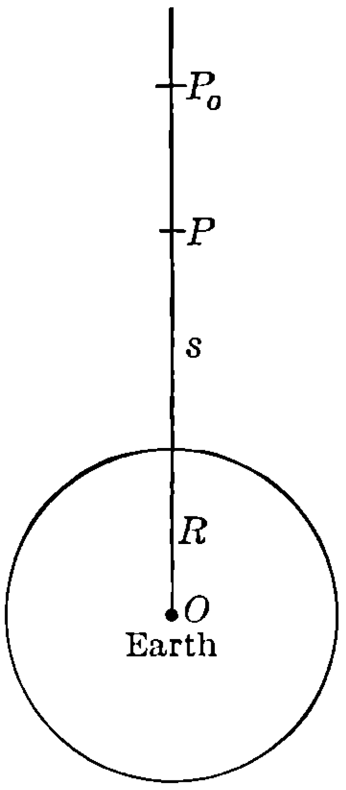


FIG. 172

$$g = \frac{k}{R^2},$$

so that $k = gR^2$. Equation (1) thus takes the specific form

$$\alpha = \frac{gR^2}{\overline{OP}^2}. \tag{2}$$

We shall now show how this equation together with the ability to solve a certain simple differential equation enables one to determine both the velocity and the position of the falling body at any time t after starting.

Let the distance between O and P at the time t as measured in the direction OP be represented by s. Then the acceleration at P will be represented by $-\frac{d^2s}{dt^2}$, the negative sign being necessary because the velocity is always decreasing in the direction along which we have agreed to measure s. Equation (2) thus takes the form

$$\frac{d^2s}{dt^2} = -\frac{gR^2}{s^2}. \tag{3}$$

This equation being of the type considered in Art. 168, we proceed to solve it in the manner there indicated.

Multiplying both members by $2\,ds$, the equation takes the form

$$\frac{d}{dt}\left(\frac{ds}{dt}\right)^2 dt = -\frac{2\,gR^2}{s^2}\,ds.$$

Integrating, we obtain

$$\left(\frac{ds}{dt}\right)^2 = \frac{2\,gR^2}{s} + c, \tag{4}$$

where the constant c is to be determined from the fact that when the body is at P_0, that is, when $s = s_0$, the value of the velocity, $\frac{ds}{dt}$, is zero. This condition when introduced in (4) gives

$$c = -\frac{2\,gR^2}{s_0},$$

so that the final form of (4) is

$$\left(\frac{ds}{dt}\right)^2 = 2\,gR^2\left(\frac{1}{s} - \frac{1}{s_0}\right), \quad \text{or} \quad \left(\frac{ds}{dt}\right)^2 = \frac{2\,gR^2}{s_0}\left(\frac{s_0 - s}{s}\right).$$

Upon extracting the square root of each member of this equation, observing that the negative sign must be taken for the radical inasmuch as the velocity, or $\frac{ds}{dt}$, continually decreases in the direction along which s is being measured, we arrive at the following result:

If a body starts from rest at a great distance s_0 from the earth's center and falls toward the earth, its velocity v when at the distance s from the center is given by the formula

$$v = -R\sqrt{\frac{2\,g}{s_0}}\sqrt{\frac{s_0 - s}{s}}, \tag{5}$$

where R is the length of the earth's radius.

In this formula let us now substitute for v its value $\frac{ds}{dt}$ and then separate the variables in the resulting equation. We thus have

$$dt = -\frac{1}{R}\sqrt{\frac{s_0}{2\,g}}\sqrt{\frac{s}{s_0 - s}}\,ds.$$

Integrating, we obtain

$$t = \frac{1}{R}\sqrt{\frac{s_0}{2\,g}}\left(\sqrt{s(s_0 - s)} + s_0 \operatorname{arc\,sin}\sqrt{\frac{s_0 - s}{s_0}}\right),$$

the constant of integration being zero since $s = s_0$ when $t = 0$. The last term can be slightly simplified by recalling the general formula

$$\operatorname{arc\,sin}\sqrt{1 - x^2} = \operatorname{arc\,cos} x.$$

When this is applied, we reach in summary the following result:

If a body starts from rest at a great distance s_0 from the earth's center and falls toward the earth, its distance s from the center at the time t is determined by the equation

$$t = \frac{1}{R}\sqrt{\frac{s_0}{2g}}\left(\sqrt{s(s_0 - s)} + s_0 \text{ arc cos } \sqrt{\frac{s}{s_0}}\right), \tag{6}$$

where R is the length of the earth's radius.

EXERCISES

1. Find the velocity with which a body arrives at the surface of the earth if it be dropped from a height equal to the earth's radius; also determine the time of falling from this height. Take $R = 4000$ miles, but note that this must be expressed in feet if we use $g = 32$.

Ans. About 5 mi. per sec.; 34 min. 50 sec., approximately.

2. Show that for bodies falling from rest near the earth's surface, that is, such that $s = R$ and $h = s_0 - s$ is very small in comparison with R, the formula (5) reduces to the familiar law $v = \sqrt{2gh}$. Compare Ex. 3, page 109.

3. Show that when s_0 is large in comparison with R while s differs but slightly from R, formulas (5) and (6) reduce approximately to

$$= -R\sqrt{\frac{2g}{s}}, \quad t = \frac{\pi}{2\sqrt{2gR}} s_0^{\frac{3}{2}}.$$

Hence find the final velocity and the time of falling of a body to the earth's surface from the moon ($s_0 = 60R$). *Ans.* $v = 7$ mi. per sec.; 116 hr.

HINT. Write the radical $\sqrt{s(s_0 - s)}$ in the form $s_0\sqrt{\frac{s}{s_0}\left(1 - \frac{s}{s_0}\right)}$.

4. By means of the formulas of Ex. 3 find the least velocity that a body projected vertically upward from the earth's surface may have in order that it shall never return. *Ans.* 7 mi. per sec.

APPENDIX

THE METHOD OF LEAST SQUARES*

172. Introduction. In many types of problems in physics, engineering, and statistics the curves with which one is called upon to deal do not have their equations known beforehand. In such cases all we know is the location of a certain number of individual points upon the curve, and these have become known only through mere observation. For example, in a laboratory experiment in physics a certain amount of air was compressed by means of a column of mercury, as shown in Fig. 173. The volumes, V, of the air corresponding to different pressures, p, were carefully noted and recorded in a number of cases, thus leading to the following table:

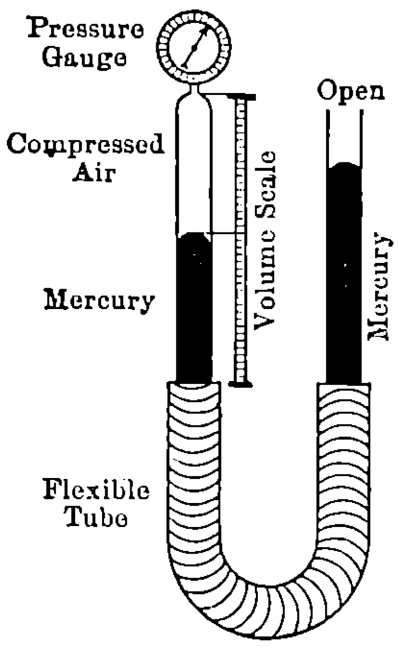


FIG. 173

PRESSURE, p	33.13	40.44	50.48	59.30	67.08	74.37
VOLUME, V	12.20	9.45	7.55	6.47	5.65	5.07

When considered graphically, as can be done by regarding the p-values as abscissas and their corresponding V-values as ordinates, this table gives rise to the six points marked + in Fig. 174, and these points, regarded as a group, evidently present a general trend or direction such as that indicated roughly by the smooth curve shown in the drawing.

* The following brief treatment of the method of least squares is essentially a reproduction of certain parts of the chapter entitled "The Elements of Curve Fitting" to be found in the author's *Brief Course in Analytic Geometry* (Holt, 1925). In the present instance the calculus has been employed wherever practicable, whereas it was naturally omitted in the chapter to which reference was just made. However, both the "method of least squares" and the "method of averages" are there treated from the standpoint.

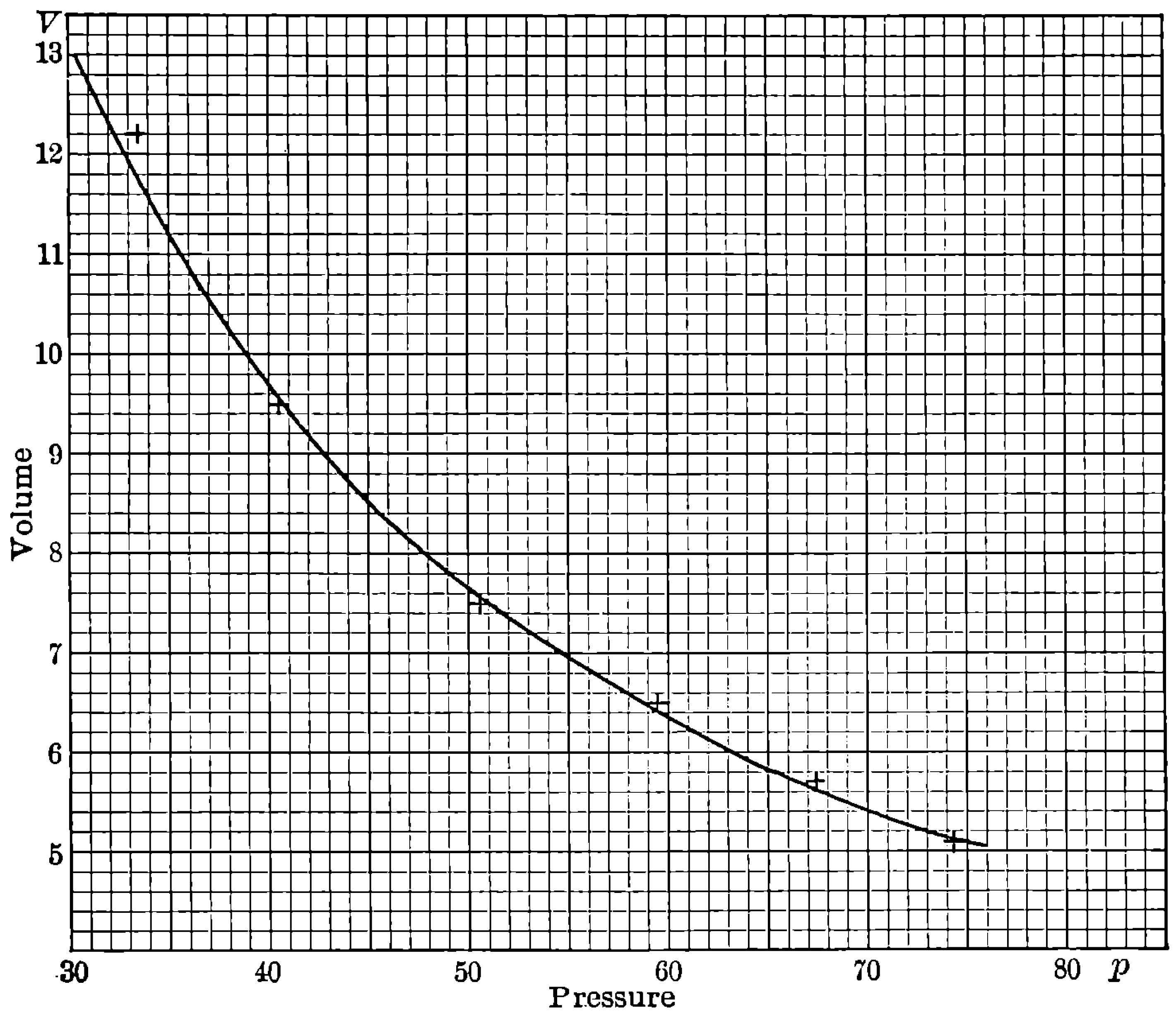


Fig. 174

It is with such curves that we shall now be concerned. We shall consider the following specific question: *Having given a group of points in the sense illustrated in Fig. 174, how can one determine the smooth curve that is best adapted to indicate their general trend?* The process of determining such a curve is called **curve-fitting.**

173. General nature of the problem. Several features of the general problem stated in the last article are to be observed. First, the greater the number of points given in the group the more apparent the general trend becomes at the outset, but as a rule the smooth curve best representing this trend will not pass directly through any one of the points themselves. On the contrary, the points will usually be distributed on either side of the curve, as illustrated in Fig. 174. Secondly, what is to be regarded as a best fit will be a matter, to a large extent, of the precise way in which we choose to define the term " best." All that we can do is to limit ourselves to some prescribed type of curve that is apparently well adapted to represent the group in question, and then, by employing certain principles, arrive at the best curve of *this type* for the purpose.

174. Definitions. A curve indicating the general trend of a set of points is called an **empirical curve.** The difference between the ordinate of a given point in a group and the ordinate of the corresponding point on an empirical curve of the group is called the **residual** of that point for the curve in question. Thus, in Fig. 175 the residuals of the points P_1, P_2, P_3, P_4, P_5 are respectively P_1Q_1, P_2Q_2, P_3Q_3, etc. When P is above the curve, as at P_2, its residual is regarded as positive; when below the curve, as at P_1, the residual is negative.

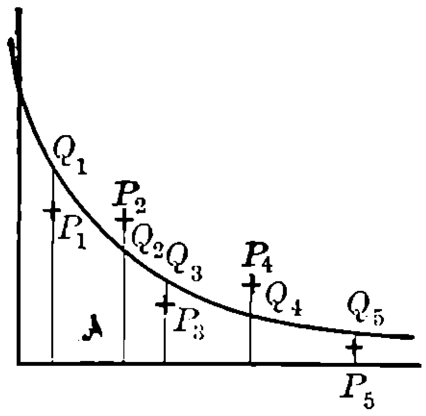


FIG. 175

175. Types of empirical curves. Restatement of the problem. The simplest type of empirical curves which one can employ is evidently that of straight lines. In this case, the equations of all the curves will be of the simple type

$$y = mx + b,$$

where m and b are constants. Similarly, whatever type of empirical curve one employs, there will correspond an equation representing this type and containing one or more constants and by attributing different values to these constants one may obtain the various curves belonging to this type. This being understood, the general problem of Art. 172 may now be narrowed to the following more specific form:

Having given a group of points and the equation of the type of empirical curve to be used in fitting it, how can the constants of the equation be determined in order to give the curve of best fit?

176. The method of least squares. Fundamental principle. Thus far we have made no attempt to define the term " best fit," but we have indicated that more than one way of doing so is possible. Of these ways the most important is the one resulting from the so-called **method of least squares,** the fundamental principle of this method being as follows:

Principle. *The empirical curve of a given type best fitting a given set of points is that one in which the constants of the type equation are so chosen as to make the sum of the squares of the residuals of the points a minimum.**

* The justification of this principle rests upon what is known in mathematics as the theory of probability and hence falls outside the scope of this elementary chapter.

With this principle assumed, we are now in position to solve, or at least to consider critically, the problem stated in Art. 175. In doing this we shall naturally treat first the case of straight-line empirical curves since they are the simplest.

177. Theorem and applications. If the method of least squares is used, the following theorem holds true:

Theorem. *The best fitting straight line for a given set of points* (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), *as determined by the method of least squares, passes through the point whose abscissa is the mean, or average, of the abscissas of the given group of points and whose ordinate is the mean of the ordinates of the same points. Moreover, this line has as its slope, m, the following value*

$$m = \frac{(x_1y_1 + x_2y_2 + \cdots + x_ny_n) - \dfrac{(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n)}{n}}{(x_1^2 + x_2^2 + \cdots + x_n^2) - \dfrac{(x_1 + x_2 + \cdots + x_n)^2}{n}}.$$

PROOF. The sum of the squares of the residuals of the given points (x_1, y_1), (x_2, y_2), $\cdots$ (x_n, y_n) with reference to any given straight line

$$y = mx + b$$

is

$$(y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \cdots + (y_n - mx_n - b)^2, \quad (1)$$

and this expression, by squaring each term as indicated and collecting like powers of b, may be written

$$\begin{aligned} nb^2 &+ 2[(x_1 + x_2 + \cdots + x_n)m - (y_1 + y_2 + \cdots + y_n)]b \\ &+ [(x_1^2 + x_2^2 + \cdots + x_n^2)m^2 - 2(x_1y_1 + x_2y_2 + \cdots + x_ny_n)m \\ &+ (y_1^2 + y_2^2 + \cdots + y_n^2)]. \end{aligned}$$

Now, by the principle of Art. 176, b is to be so chosen that this quadratic expression in b shall be a minimum. Hence, by Art. 37, b is to be determined so that the derivative of this expression with respect to b shall be equal to zero. We thus obtain for b the equation

$$nb + (x_1 + x_2 + \cdots + x_n)m - (y_1 + y_2 + \cdots + y_n) = 0, \quad (2)$$

or, dividing both members by n,

$$\left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right) = m\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) + b.$$

This equation indicates that the point

$$\left(\frac{x_1 + x_2 + \quad + x_n}{n}, \frac{y_1 + y_2 + \quad + y_n}{n}\right)$$

lies on the best line, $y = mx + b$, and thus the first part of the theorem becomes established.

Secondly, if we arrange the expression (1) in descending powers of m, it becomes

$$(x_1^2 + x_2^2 + \quad + x_n^2)m^2 + 2[(x_1 + x_2 + \quad + x_n)b$$
$$-(x_1y_1 + x_2y_2 + \quad + x_ny_n)]m + [nb^2 - 2(y_1 + y_2 + \quad + y_n)b$$
$$+(y_1^2 + y_2^2 + \quad + y_n^2)],$$

and, by again applying the principle of Art. 176 together with the elementary results in Art. 37 we may say that m is to be so determined in the present instance that the derivative of this expression with respect to m shall be equal to zero. Hence, differentiating, the following equation results:

$$(x_1^2 + x_2^2 + \quad + x_n^2)m + (x_1 + x_2 + \quad + x_n)b$$
$$-(x_1y_1 + x_2y_2 + \quad + x_ny_n) = 0. \qquad (3)$$

By multiplying both members of (2) by

$$\frac{x_1 + x_2 + \quad + x_n}{n},$$

then subtracting the result member by member from (3), and solving the resulting equation for m, we find it to be as stated in the theorem, thus completing the proof.

Example. The electrical resistance of a copper wire increases as the temperature to which it is heated is increased. In a particular experiment the resistance was measured at five different temperatures. The results are shown in the following table where the resistance is measured in ohms and the temperature in degrees centigrade.

TEMPERATURE, t	21.8	35.5	47.1	60.5	68.5
RESISTANCE, R	58.30	61.39	63.90	67.01	69.53

Thus, Fig. 176 results, and it appears at once that the general trend of the points is in a straight-line direction. Determine by the method of least squares the best fitting line for this table.

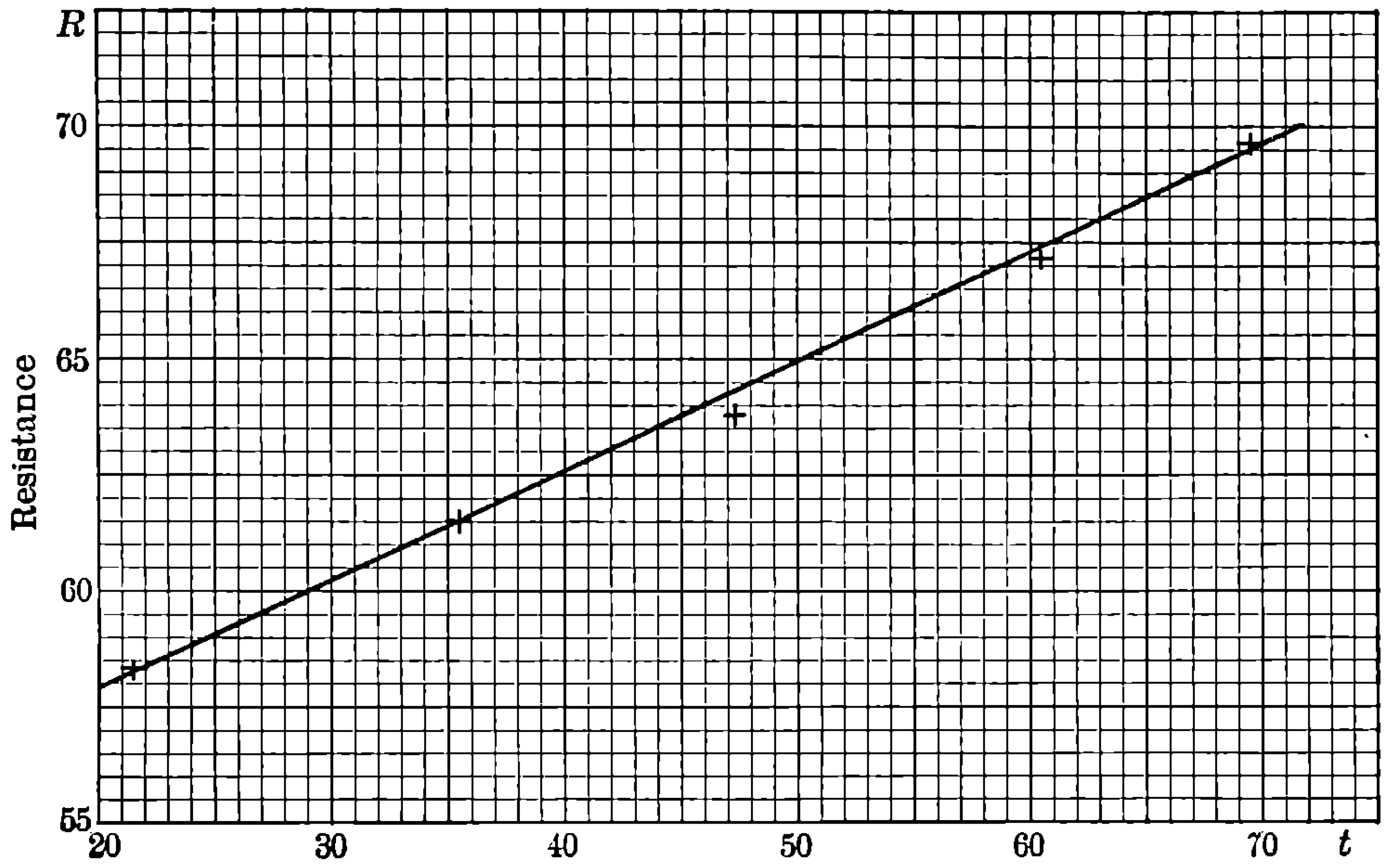


FIG. 176

SOLUTION. It is desirable first to arrange the values of the temperature t and of the resistance R (tabular values) in vertical columns rather than horizontal rows, appending also the values of t^2 and tR as in the following table:

t	R	t^2	tR
21.8	58.30	475.24	1270.940
35.5	61.39	1260.25	2179.345
47.1	63.90	2218.41	3009.690
60.5	67.01	3660.25	4054.105
68.5	69.53	4692.25	4762.805
233.4	320.13	12306.40	15276.885

(4)

The single point upon the desired line, as furnished by the theorem, is thus the one whose coördinates are respectively

$$\frac{233.4}{5} = 46.68 \quad \text{and} \quad \frac{320.13}{5} = 64.026.$$

As to the slope, m, of the desired line as determined by the theorem, the numbers for its determination being readily at hand in table (4), we have for m the value

$$m = \frac{15276.885 - \dfrac{233.4 \times 320.13}{5}}{12306.40 - \dfrac{(233.4)^2}{5}} = \frac{333.217}{1411.29} = .2361.$$

Hence the desired line passes through the point (46.68, 64.026) and has the slope .2361. Its equation is therefore

$$R - 64.026 = .2361(t - 46.68)$$

or

$$R = .2361\,t + 53.01.$$

EXERCISES

1. In order that a given amount of gas shall maintain a constant pressure when different amounts of heat are applied to it, it is necessary that its volume be changed. The following table records the results in an experiment of this kind, t denoting temperature in degrees centigrade and v the corresponding volume in cubic centimeters:

t	20	40	60	80	100
	255	271	290	307	326

Find, by the method of least squares, the best fitting straight line $v = mt + b$ corresponding to this table. *Ans.* $v = 236.4 + .89\,t.$

2. Use your result for Ex. 1 to determine the volume at intervals of 10° from 0° to 100° If your line indicated the correct law for all temperatures, at what temperature would the volume vanish?

3. In a certain electric circuit, different voltages, e, were applied to overcome the resistance. For each the resulting current, i, expressed in amperes flowing through the circuit was measured, leading to the following table:

	2.1	4.6	6.3	8.4	10.1
	0.172	0.381	0.523	0.702	0.842

Find, by the method of least squares, the best fitting line $i = me + b$ corresponding to this table. *Ans.* $i = .0839\,e - .0045.$

4. Use your result for Ex. 3 to determine the amperes of current corresponding to the given voltages, and compare your results with those of the table for these same voltages. What is the extreme discrepancy thus seen to be present? By means of your result for Ex. 3 answer also the following questions: (*a*) What voltage will produce a current of 1 ampere? (*b*) If the resistance is protected by a $1\frac{1}{2}$ ampere fuse, what is the maximum voltage that can be supplied before destroying the fuse?

5. Find the straight line best fitting the points (5, 8), (8, 13), (13, 21), (21, 34). By how much does the line miss the point (34, 55)?

178. Auxiliary theorem. An unfortunate feature of curve-fitting is the tedious arithmetical calculations that have to be performed in evaluating the constants of the desired equation. It is desirable, therefore, to take advantage of every possible device for shortening this part of the work. Thus, the following auxiliary theorem frequently proves useful:

Auxiliary Theorem. *The origin and the units of measure used for the abscissas and ordinates, respectively, can be changed without affecting the best line, except to refer it to new axes and new units.*

This theorem allows us, in particular, to subtract an arbitrary constant from each abscissa appearing in any given table and to divide the differences by an arbitrary constant before applying the method of least squares. The ordinates may be treated in a similar manner. The advantage of such changes lies in the fact that we can thus reduce, in many cases, the size of the numbers upon which our computations are based.

PROOF. Suppose the given points are (x_1, y_1), (x_2, y_2), $\quad$ (x_n, y_n). Let the origin be changed to the point (h, k) and let u units of x be equal to one unit of x' and let v units of y be equal to one unit of y' Then we shall have as the equations of transformation

$$x = ux' + h, \quad y = vy' + k, \tag{1}$$

while the equation of any line, as $y = mx + b$, becomes

$$vy' + k = mux' + mh + b. \tag{2}$$

Now, if the line $y = mx + b$ is the best line, we know by the theorem of Art. 177 that the point

$$\left(\frac{x_1 + x_2 + \quad + x_n}{n}, \frac{y_1 + y_2 + \quad + y_n}{n}\right)$$

lies upon it. But, using (1), we here have

$$\frac{x_1 + x_2 + \quad + x_n}{n} = \frac{u(x_1' + x_2' + \quad + x_n') + nh}{n}$$

$$= u\left(\frac{x_1' + x_2' + \quad + x_n'}{n}\right) + h,$$

$$\frac{y_1 + y_2 + \quad + y_n}{n} = \frac{v(y_1' + y_2' + \quad + y_n') + nk}{n}$$

$$= v\left(\frac{y_1' + y_2' + \quad + y_n'}{n}\right) + k,$$

so that for the best line we must have

$$v\left(\frac{y_1' + y_2' + \quad + y_n'}{n}\right) + k$$

$$= mu\left(\frac{x_1' + x_2' + \quad + x_n'}{n}\right) + mh + b.$$

Thus the point

$$\left(\frac{x_1' + x_2' + \quad + x_n'}{n}, \frac{y_1' + y_2' + \quad + y_n'}{n}\right) \tag{3}$$

must lie upon (2). But the point (3) is seen to be no other than the one we obtain by applying the method of least squares at the outset to the data referred to the new origin and units.

It remains to show that the slope of the best line as computed from the modified table is equal to the slope of the line (2), the latter being easily seen to be $\frac{mu}{v}$.

Since, by (1), we have $x = ux' + h$, it follows that

$$(x_1y_1 + x_2y_2 + \quad + x_ny_n) - \frac{(x_1 + x_2 + \quad + x_n)(y_1 + y_2 + \quad + y_n)}{n}$$

$$= [(ux_1' + h)y_1 + (ux_2' + h)y_2 + \quad + (ux_n' + h)y_n]$$
$$- \frac{[(ux_1' + h) + (ux_2' + h) + \quad + (ux_n' + h)](y_1 + y_2 + \quad + y_n)}{n}$$

$$= u(x_1'y_1 + x_2'y_2 + \quad + x_n'y_n) + h(y_1 + y_2 + \quad + y_n)$$
$$- \frac{u(x_1' + x_2' + \quad + x_n')(y_1 + y_2 + \quad + y_n)}{n}$$
$$- h(y_1 + y_2 + \quad + y_n)$$

$$= u\left[(x_1'y_1 + x_2'y_2 + \quad + x_n'y_n) - \frac{(x_1' + x_2' + \quad + x_n')(y_1 + y_2 + \quad + y_n)}{n}\right].$$

Similarly, since $y = vy' + k$, the last expression may be written in the form

$$uv\left[(x_1'y_1' + x_2'y_2' + \quad + x_n'y_n') - \frac{(x_1' + x_2' + \quad + x_n')(y_1' + y_2' + \quad + y_n')}{n}\right].$$

In like manner we can show that

$$x_1^2 + x_2^2 + \quad + x_n^2 - \frac{(x_1 + x_2 + \quad + x_n)^2}{n}$$
$$= u^2\left[(x_1'^2 + x_2'^2 + \quad + x_n'^2) - \frac{(x_1' + x_2' + \quad + x_n')^2}{n}\right].$$

Hence, for the slope, $\frac{mu}{v}$, of the best line (2) we have, upon using the value of m given by the theorem of Art. 177,

$$\frac{mu}{v} = \frac{u}{v} m$$
$$= \frac{u}{v} \cdot \frac{x_1y_1 + x_2y_2 + \quad + x_ny_n - \frac{(x_1 + x_2 + \quad + x_n)(y_1 + y_2 + \quad + y_n)}{n}}{x_1^2 + x_2^2 + \quad + x_n^2 - \frac{(x_1 + x_2 + \quad + x_n)^2}{n}}$$
$$= \frac{x_1'y_1' + x_2'y_2' + \quad + x_n'y_n' - \frac{(x_1' + x_2' + \quad + x_n')(y_1' + y_2' + \quad + y_n')}{n}}{x_1'^2 + x_2'^2 + \quad + x_n'^2 - \frac{(x_1' + x_2' + \quad + x_n')^2}{n}},$$

which is seen to be the slope, as computed by the same theorem of Art. 177, directly from the modified table.

Thus, the proof becomes complete.

Example. The following table gives the number of millions of population in France at each census from 1806 to 1866:

t = Year	1806	1821	1831	1836	1841	1846	1851	1856	1861	1866
P = Millions Population	29.11	30.46	32.57	33.54	34.23	35.40	35.78	36.04	37.39	38.07

To determine, by the method of least squares, a straight line well adapted to this table.

Solution. The graph of the table, as shown on the next page, reveals a straight-line trend, thus making the problem a reasonable one at the outset. This being assured, we next apply the auxiliary theorem of this article in order to reduce the size of the numbers involved.

Let us take 1836 as the new origin of time and five years as the new unit of time; also, let us take twenty-nine million as the new origin of population and one million as the new unit of population. If (t, P) be taken as the coördinates of any point in the original system, and if (t', P') be regarded as coördinates of the same point in the new system, the equations of transformation (1) then assume the forms

$$t = 5\,t' + 1836, \quad P = 1{,}000{,}000\,P' + 29{,}000{,}000. \qquad (4)$$

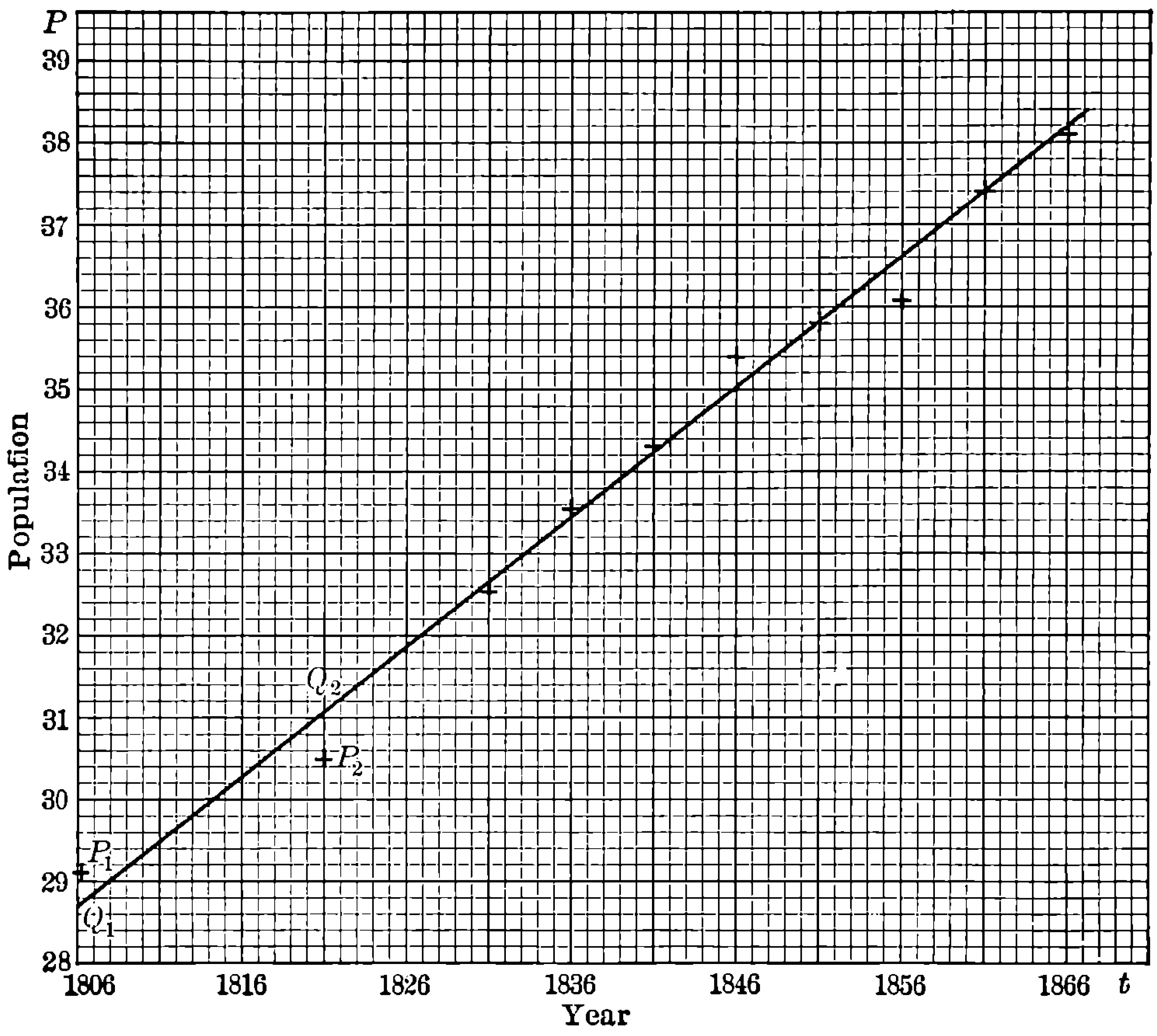


FIG. 177

Letting t and P take on in (4) the special values indicated in the table and writing the corresponding modified table in the extended form (4) of Art. 177, we have

t'	P'	t'^2	$t'P'$
− 6	.11	36	− .66
− 3	1.46	9	− 4.38
− 1	3.57	1	− 3.57
0	4.54	0	0
1	5.23	1	5.23
2	6.40	4	12.80
3	6.78	9	20.34
4	7.04	16	28.16
5	8.39	25	41.95
6	9.07	36	54.42
11	52.59	137	154.29

Applying the theorem of Art. 177 to this modified table, we see that the point upon the best line thereby determined has its abscissa and ordinate equal respectively to

$$\frac{11}{10} = 1.1 \quad \text{and} \quad \frac{52.59}{10} = 5.259,$$

while, for the slope m we obtain by the same theorem

$$m = \frac{154.29 - \dfrac{11 \times 52.59}{10}}{137 - \dfrac{11^2}{10}} = \frac{96.441}{124.90} = .7722.$$

Hence, the desired line passes through the point (1.1, 5.259) and has a slope equal to .7722. The equation of the desired line, by analytic geometry, is therefore

$$P' - 5.259 = .7722(t' - 1.1) \text{ or } P' = .7722\,t' + 4.410. \qquad (5)$$

The passage from (5) to the final answer is to be obtained by substituting

$$\frac{t - 1836}{5} \text{ for } t' \quad \text{and} \quad \frac{P - 29{,}000{,}000}{1{,}000{,}000} \text{ for } P'.$$

Thus we obtain after reductions as the required line

$$P = 1{,}000{,}000\,(.1545\,t - 250.08).$$

EXERCISES

1. In a certain coil of wire the electrical resistance, r, measured in ohms, corresponding to different temperatures, t, measured in degrees centigrade, was recorded and resulted in the following table:

t	10	30	50	70	90
	10.121	10.689	11.270	11.838	12.413

Find the best fitting line $r = mt + b$ corresponding to this table, using the method of least squares. *Ans.* $r = 9.834 + .02864\,t$.

2. The lengths, l, attained by a certain coiled spring made of steel wire, corresponding to different weights, w, supported by the spring were as shown in the following table, where lengths are measured in centimeters and weights in grams:

	100	200	300	400	500	600	700
l	92.5	94.7	96.8	98.9	100.8	102.9	104.9

Find the best fitting line $l = mw + b$ corresponding to this table, using the method of least squares with new origin at (400, 100), and with (100 gms, 1 cm) as new units. *Ans.* $l = 90.56 + .02057\,w$.

179. Parabolic, hyperbolic, and exponential empirical curves. The foregoing applications of the method of least squares have been confined to the determination of the best fitting straight line corresponding to a given table of data. Naturally, the best fit for a given table will not always be of a straight-line nature. In particular, the points corresponding to the data may have, and frequently do have, the trend of a parabola, hyperbola, or an exponential curve. The method of least squares may here be applied, as in the case of a straight-line trend, to secure the curve of best fit belonging to any one basic variety, such as parabolic. For a discussion of this, with examples and applications, the student is referred to the author's *Brief Course in Analytic Geometry*, Arts. 98–104.

MISCELLANEOUS EXERCISES

1. When an observer moves towards a source of sound the pitch p increases as the velocity v of the observer increases. If the relation is as indicated in the following table, find the best fitting straight line $p = mv + b$ which expresses p in terms of v, using the method of least squares.

	15	20	25	30	35	40
p	268	272	275	279	284	287

2. The weight of common salt which will dissolve in 100 g. of water is as shown in the following table:

t	0	10	30	50	100
	35.7	35.8	36.3	37.0	39.8

Find by the method of least squares the best fitting straight line $w = mt + b$ for this data.

3. The following table gives the average weights (w lbs.) of men of various heights (h in.):

h	61	63	65	67	69	71
w	123.5	132.4	140.1	147.2	157.3	166.7

Find by the method of least squares the best fitting straight line $w = mh + b$ for this data.

TABLES

I. TABLES OF INDEFINITE INTEGRALS

A. Forms Involving $a + bx$

1. $$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x} + C.$$

2. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x} + C.$$

3. $$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right] + C.$$

4. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a + bx - 2a\log(a+bx) - \frac{a^2}{a+bx}\right] + C.$$

5. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x} + C.$$

6. $$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C.$$

7. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C.$$

8. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx} + C.$$

9. $$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx} + C.$$

10. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} + C, \text{ when } a > 0.$$

11. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\arctan\sqrt{\frac{a+bx}{-a}} + C, \text{ when } a < 0.$$

12. $$\int \frac{dx}{x^2\sqrt{a+bx}} = \frac{-\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}} + C.$$

13 $$\int \frac{\sqrt{a+bx}\,dx}{x} = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}} + C.$$

B. Forms Involving $\sqrt{x^2 + a^2}$

14. $\int (x^2 + a^2)^{\frac{1}{2}} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2}) + C.$

15. $\int (x^2 + a^2)^{\frac{3}{2}} dx = \frac{x}{8}(2x^2 + 5a^2)\sqrt{x^2 + a^2} + \frac{3a^4}{8}\log(x + \sqrt{x^2 + a^2}) + C.$

16. $\int x^2(x^2 + a^2)^{\frac{1}{2}} dx = \frac{x}{8}(2x^2 + a^2)\sqrt{x^2 + a^2} - \frac{a^4}{8}\log(x + \sqrt{x^2 + a^2}) + C.$

17. $\int \frac{dx}{(x^2 + a^2)^{\frac{1}{2}}} = \log(x + \sqrt{x^2 + a^2}) + C.$

18. $\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2 + a^2}} + C.$

19. $\int \frac{x\,dx}{(x^2 + a^2)^{\frac{1}{2}}} = \sqrt{x^2 + a^2} + C.$

20. $\int \frac{x^2\,dx}{(x^2 + a^2)^{\frac{1}{2}}} = \frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2}) + C.$

21. $\int \frac{x^2\,dx}{(x^2 + a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2 + a^2}} + \log(x + \sqrt{x^2 + a^2}) + C.$

22. $\int \frac{dx}{x(x^2 + a^2)^{\frac{1}{2}}} = \frac{1}{a}\log\frac{x}{a + \sqrt{x^2 + a^2}} + C.$

23. $\int \frac{dx}{x^2(x^2 + a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2 + a^2}}{a^2x} + C.$

24. $\int \frac{dx}{x^3(x^2 + a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2 + a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{a + \sqrt{x^2 + a^2}}{x} + C.$

25. $\int \frac{(x^2 + a^2)^{\frac{1}{2}}\,dx}{x} = \sqrt{a^2 + x^2} - a\log\frac{a + \sqrt{a^2 + x^2}}{x} + C.$

26. $\int \frac{(x^2 + a^2)^{\frac{1}{2}}\,dx}{x^2} = -\frac{\sqrt{x^2 + a^2}}{x} + \log(x + \sqrt{x^2 + a^2}) + C.$

C. Forms Involving $\sqrt{x^2 - a^2}$

27. $\int (x^2 - a^2)^{\frac{1}{2}} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2 - a^2}) + C.$

28. $\int (x^2 - a^2)^{\frac{3}{2}} dx = \frac{x}{8}(2x^2 - 5a^2)\sqrt{x^2 - a^2} + \frac{3a^4}{8}\log(x + \sqrt{x^2 - a^2}) + C.$

29. $\int x^2(x^2 - a^2)^{\frac{1}{2}} dx = \frac{x}{8}(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8}\log(x + \sqrt{x^2 - a^2}) + C.$

30. $\displaystyle\int \frac{dx}{(x^2 - a^2)^{\frac{1}{2}}} = \log(x + \sqrt{x^2 - a^2}) + C.$

31. $\displaystyle\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}} = -\frac{x}{a^2\sqrt{x^2 - a^2}} + C.$ 32. $\displaystyle\int \frac{x\,dx}{(x^2 - a^2)^{\frac{1}{2}}} = \sqrt{x^2 - a^2} + C.$

33. $\displaystyle\int \frac{x^2\,dx}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2 - a^2}) + C.$

34. $\displaystyle\int \frac{x^2\,dx}{(x^2 - a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2 - a^2}} + \log(x + \sqrt{x^2 - a^2}) + C.$

35. $\displaystyle\int \frac{dx}{x(x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{a}\operatorname{arc\,sec}\frac{x}{a} + C.$

36. $\displaystyle\int \frac{dx}{x^2(x^2 - a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2 - a^2}}{a^2x} + C.$

37. $\displaystyle\int \frac{dx}{x^3(x^2 - a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2 - a^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\operatorname{arc\,sec}\frac{x}{a} + C.$

38. $\displaystyle\int \frac{(x^2 - a^2)^{\frac{1}{2}}\,dx}{x} = \sqrt{x^2 - a^2} - a\operatorname{arc\,cos}\frac{a}{x} + C.$

39. $\displaystyle\int \frac{(x^2 - a^2)^{\frac{1}{2}}\,dx}{x^2} = -\frac{\sqrt{x^2 - a^2}}{x} + \log(x + \sqrt{x^2 - a^2}) + C.$

D. Forms Involving $\sqrt{a^2 - x^2}$

40. $\displaystyle\int (a^2 - x^2)^{\frac{1}{2}}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\operatorname{arc\,sin}\frac{x}{a} + C.$

41. $\displaystyle\int (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{x}{8}(5\,a^2 - 2\,x^2)\sqrt{a^2 - x^2} + \frac{3\,a^4}{8}\operatorname{arc\,sin}\frac{x}{a} + C.$

42. $\displaystyle\int x^2(a^2 - x^2)^{\frac{1}{2}}\,dx = \frac{x}{8}(2\,x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\operatorname{arc\,sin}\frac{x}{a} + C.$

43. $\displaystyle\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \operatorname{arc\,sin}\frac{x}{a} + C.$ 44. $\displaystyle\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + C.$

45. $\displaystyle\int \frac{x\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\sqrt{a^2 - x^2} + C.$

46. $\displaystyle\int \frac{x^2\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\operatorname{arc\,sin}\frac{x}{a} + C.$

47. $\displaystyle\int \frac{x^2\,dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2 - x^2}} - \operatorname{arc\,sin}\frac{x}{a} + C.$

48. $\displaystyle\int \frac{dx}{x(a^2 - x^2)^{\frac{1}{2}}} = \frac{1}{a} \log \frac{x}{a + \sqrt{a^2 - x^2}} + C.$

49. $\displaystyle\int \frac{dx}{x^2(a^2 - x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$

50. $\displaystyle\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x} dx = \sqrt{a^2 - x^2} - a \log \frac{a + \sqrt{a^2 - x^2}}{x} + C.$

51. $\displaystyle\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \text{arc sin} \frac{x}{a} + C.$

E. Forms Involving $\sqrt{2\,ax - x^2}$, $\sqrt{2\,ax + x^2}$

52. $\displaystyle\int \sqrt{2\,ax - x^2}\, dx = \frac{x - a}{2} \sqrt{2\,ax - x^2} + \frac{a^2}{2} \text{arc vers} \frac{x}{a} + C.$

53. $\displaystyle\int \frac{dx}{\sqrt{2\,ax - x^2}} = \text{arc vers} \frac{x}{a} + C.$

54. $\displaystyle\int x\sqrt{2\,ax - x^2}\, dx = -\frac{3\,a^2 + ax - 2\,x^2}{6} \sqrt{2\,ax - x^2} + \frac{a^3}{2} \text{arc vers} \frac{x}{a} + C.$

55. $\displaystyle\int \frac{dx}{x\sqrt{2\,ax - x^2}} = -\frac{\sqrt{2\,ax - x^2}}{ax} + C.$

56. $\displaystyle\int \frac{x\,dx}{\sqrt{2\,ax - x^2}} = -\sqrt{2\,ax - x^2} + a\, \text{arc vers} \frac{x}{a} + C.$

57. $\displaystyle\int \frac{x^2\,dx}{\sqrt{2\,ax - x^2}} = -\frac{x + 3\,a}{2} \sqrt{2\,ax - x^2} + \frac{3}{2} a^2 \text{arc vers} \frac{x}{a} + C.$

58. $\displaystyle\int \frac{\sqrt{2\,ax - x^2}}{x} dx = \sqrt{2\,ax - x^2} + a\, \text{arc vers} \frac{x}{a} + C.$

59. $\displaystyle\int \frac{\sqrt{2\,ax - x^2}}{x^2} dx = -\frac{2\sqrt{2\,ax - x^2}}{x} - \text{arc vers} \frac{x}{a} + C.$

60. $\displaystyle\int \frac{\sqrt{2\,ax - x^2}}{x^3} dx = -\frac{(2\,ax - x^2)^{\frac{3}{2}}}{3\,ax^3} + C.$

61. $\displaystyle\int \frac{dx}{(2\,ax - x^2)^{\frac{3}{2}}} = \frac{x - a}{a^2\sqrt{2\,ax - x^2}} + C.$

62. $\displaystyle\int \frac{dx}{\sqrt{2\,ax + x^2}} = \log (x + a + \sqrt{2\,ax + x^2}) + C.$

F. Forms Involving $a + bx \pm cx^2$, where $c > 0$

63. $$\int \frac{dx}{a + bx + cx^2} = \frac{2}{\sqrt{4\,ac - b^2}} \arctan \frac{2\,cx + b}{\sqrt{4\,ac - b2}} + C, \text{ when } b^2 < 4\,ac.$$

64. $$\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4\,ac}} \log \frac{2\,cx + b - \sqrt{b^2 - 4\,ac}}{2\,cx + b + \sqrt{b^2 - 4\,ac}} + C,$$
when $b^2 > 4\,ac$.

65. $$\int \frac{dx}{a + bx - cx^2} = \frac{1}{\sqrt{b^2 + 4\,ac}} \log \frac{\sqrt{b^2 + 4\,ac} + 2\,cx - b}{\sqrt{b^2 + 4\,ac} - 2\,cx + b} + C.$$

66. $$\int \sqrt{a + bx + cx^2}\, dx = \frac{2\,cx + b}{4\,c}\sqrt{a + bx + cx^2} - \frac{b^2 - 4\,ac}{8\,c^{\frac{3}{2}}} \log\left(2\,cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}\right) + C.$$

67. $$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log\left(2\,cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}\right) + C.$$

68. $$\int \frac{dx}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \arcsin \frac{2\,cx - b}{\sqrt{b^2 + 4\,ac}} + C.$$

69. $$\int \sqrt{a + bx - cx^2}\, dx = \frac{2\,cx - b}{4\,c}\sqrt{a + bx - cx^2} + \frac{b^2 + 4\,ac}{8\,c^{\frac{3}{2}}} \arcsin \frac{2\,cx - b}{\sqrt{b^2 + 4\,ac}} + C.$$

70. $$\int \frac{x\,dx}{\sqrt{a + bx + cx^2}} = \frac{\sqrt{a + bx + cx^2}}{c} - \frac{b}{2\,c^{\frac{3}{2}}} \log\left(2\,cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}\right) + C.$$

71. $$\int \frac{x\,dx}{\sqrt{a + bx - cx^2}} = -\frac{\sqrt{a + bx - cx^2}}{c} + \frac{b}{2\,c^{\frac{3}{2}}} \arcsin \frac{2\,cx - b}{\sqrt{b^2 + 4\,ac}} + C.$$

G. Forms Involving the Trigonometric Functions

72. $$\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{4}\sin 2\,x + C.$$

73. $$\int \cos^2 x\, dx = \frac{x}{2} + \frac{1}{4}\sin 2\,x + C.$$

74. $$\int \sin mx \sin nx\, dx = -\frac{\sin (m + n)x}{2(m + n)} + \frac{\sin (m - n)x}{2(m - n)} + C.$$

75. $\int \cos mx \cos nx\, dx = \frac{\sin (m+n)x}{2(m+n)} + \frac{\sin (m-n)x}{2(m-n)} + C.$

76. $\int \sin mx \cos nx\, dx = -\frac{\cos (m+n)x}{2(m+n)} - \frac{\cos (m-n)x}{2(m-n)} + C.$

77. $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \arctan\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right) + C$, when $a > b$.

78. $\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}} + C$, when $a < b$.

79. $\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \arctan \frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}} + C$, when $a > b$.

80. $\int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{a\tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\frac{x}{2} + b + \sqrt{b^2 - a^2}} + C$, when $a < b$.

81. $\int \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \arctan\left(\frac{b\tan x}{a}\right) + C.$

82. $\int e^{ax} \sin nx\, dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2 + n^2} + C.$

83. $\int e^{ax} \cos nx\, dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2 + n^2} + C.$

H. Definite Integrals

Repeated use of

84. $$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta\, d\theta = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta\, d\theta$$
$$= \frac{n-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^{n-2} \theta\, d\theta,$$

where m and n are positive integers or zero, makes this frequently occurring integral depend on one of the following:

85. $\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta = 1.$

86. $\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta\, d\theta = \frac{1}{2}.$

87. $\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.$

II. FORMULAS OF TRIGONOMETRY

Whatever the angle A, the following formulas hold true:

$$\csc A = \frac{1}{\sin A}, \quad \cot A = \frac{1}{\tan A}, \quad \sec A = \frac{1}{\cos A}. \tag{1}$$

$$\sin^2 A + \cos^2 A = 1. \tag{2}$$

$$\sec^2 A = 1 + \tan^2 A. \tag{3}$$

$$\csc^2 A = 1 + \cot^2 A. \tag{4}$$

$$\tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{\cos A}{\sin A}. \tag{5}$$

$$\sin 2A = 2 \sin A \cos A. \tag{6}$$

$$\cos 2A = \cos^2 A - \sin^2 A. \tag{7}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \tag{8}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A, \quad \cos 3A = 4 \cos^3 A - 3 \cos A. \tag{9}$$

$$\sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}, \quad \cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}. \tag{10}$$

Also the following formulas hold for any two angles A, B:

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B. \tag{11}$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B. \tag{12}$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}. \tag{13}$$

III. VALUES OF THE TRIGONOMETRIC FUNCTIONS OF CERTAIN SPECIAL ANGLES

The values of the trigonometric functions of 0°, 30°, 45°, 60°, and 90° may be expressed as in the following table:

SPECIAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

ANGLE	sin	cos	tan	cot	sec	csc
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	1	0	∞	0	∞	1

IV. RADIAN MEASUREMENT OF AN ANGLE

In surveying and other immediate applications of trigonometry the customary unit of angular measurement is the degree, which is defined as the angle subtended at the center of a circle by an arc whose length is $\frac{1}{360}$ of the circumference. In the calculus, however, and in all theoretic considerations the unit of angular measurement is taken as the **radian** which is defined as the angle subtended at the center of a circle by an arc equal in length to the radius.

From these definitions it follows by taking the radius for simplicity as 1 unit that

$$2\pi \textit{ radians} = 360°, \tag{1}$$

each member of this equation being the entire length of the circumference. Therefore

$$1 \text{ radian} = \frac{360°}{2\pi} = \frac{360°}{6.2832}, \text{ approximately.}$$

Hence,

$$\mathbf{1} \textit{ radian} = \mathbf{57.29578°} = \mathbf{57°\ 17'\ 44.8''}, \text{ approximately.}$$

In practice, when radian measurement is being used, the word radian is suppressed.

Thus, the angle π means the angle π radians, which by (1) is equivalent to 180°. Similarly, the angle $\frac{\pi}{3} = \frac{\pi}{3}$ radians $= 60°$; $\frac{3\pi}{4} = 135°$, etc.

V. PROOF THAT $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ IN CASE x IS MEASURED IN RADIANS

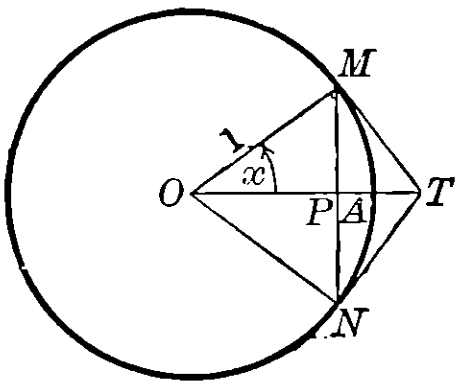


FIG. 178

Draw a circle of unit radius and at its center O lay off an angle $AOM = x$. At M draw the perpendicular MP to OA and extend it to meet the circle again at N. Draw tangents at M and N meeting OA prolonged at T Then $MP = \sin x$, $MT = \tan x$, and arc $MA = x$. Moreover, by measuring x in radians, arc MA becomes measured in the same unit as MP and MT, thus making it possible to compare x with these magnitudes and hence in particular to study the proposed ratio $\frac{\sin x}{x}$ as $x \to 0$.

Now from geometry, we have

$$MPN < \text{arc}\, MAN < MT + NT$$

or

$$2 \sin x < 2x < 2 \tan x.$$

Dividing through by 2 sin x, we thus have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Suppose now that $x \rightarrow 0$. The foregoing inequality holds true at all stages as x diminishes. At the same time

$$\lim_{x \rightarrow 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1.$$

The inequality is thus seen to be impossible unless

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

which must therefore be the case, as was to be proved.

VI. TRIGONOMETRIC TABLES

The tables which follow at this point on pages 362–366 show the values to four places of decimals of the sine, cosine, tangent, and cotangent of angles at intervals of 10 minutes ranging from 0° to 90° The value of each angle in radians is also shown.

Radians	Degrees	Sines	Cosines	Tangents	Cotangents	Degrees	Radians
0.0000	0° 00′	.0000	1.0000	.0000	∞	90° 00′	1.5708
0.0029	10	.0029	1.0000	.0029	343.77	50	1.5679
0.0058	20	.0058	1.0000	.0058	171.89	40	1.5650
0.0087	30	.0087	1.0000	.0087	114.59	30	1.5621
0.0116	40	.0116	.9999	.0116	85.940	20	1.5592
0.0145	50	.0145	.9999	.0145	68.750	10	1.5563
0.0175	1° 00′	.0175	.9998	.0175	57.290	89° 00′	1.5533
0.0204	10	.0204	.9998	.0204	49.104	50	1.5504
0.0233	20	.0233	.9997	.0233	42.964	40	1.5475
0.0262	30	.0262	.9997	.0262	38.188	30	1.5446
0.0291	40	.0291	.9996	.0291	34.368	20	1.5417
0.0320	50	.0320	.9995	.0320	31.242	10	1.5388
0.0349	2° 00′	.0349	.9994	.0349	28.636	88° 00′	1.5359
0.0378	10	.0378	.9993	.0378	26.432	50	1.5330
0.0407	20	.0407	.9992	.0407	24.542	40	1.5301
0.0436	30	.0436	.9990	.0437	22.904	30	1.5272
0.0465	40	.0465	.9989	.0466	21.470	20	1.5243
0.0495	50	.0494	.9988	.0495	20.206	10	1.5213
0.0524	3° 00′	.0523	.9986	.0524	19.081	87° 00′	1.5184
0.0553	10	.0552	.9985	.0553	18.075	50	1.5155
0.0582	20	.0581	.9983	.0582	17.169	40	1.5126
0.0611	30	.0610	.9981	.0612	16.350	30	1.5097
0.0640	40	.0640	.9980	.0641	15.605	20	1.5068
0.0669	50	.0069	.9978	.0670	14.924	10	1.5039
0.0698	4° 00′	.0698	.9976	.0699	14.301	86° 00′	1.5010
0.0727	10	.0727	.9974	.0729	13.727	50	1.4981
0.0756	20	.0756	.9971	.0758	13.197	40	1.4952
0.0785	30	.0785	.9969	.0787	12.706	30	1.4923
0.0814	40	.0814	.9967	.0816	12.251	20	1.4893
0.0844	50	.0843	.9964	.0846	11.826	10	1.4864
0.0873	5° 00′	.0872	.9962	.0875	11.430	85° 00′	1.4835
0.0902	10	.0901	.9959	.0904	11.059	50	1.4806
0.0931	20	.0929	.9957	.0934	10.712	40	1.4777
0.0960	30	.0958	.9954	.0963	10.385	30	1.4748
0.0989	40	.0987	.9951	.0992	10.078	20	1.4719
0.1018	50	.1016	.9948	.1022	9.7882	10	1.4690
0.1047	6° 00′	.1045	.9945	.1051	9.5144	84° 00′	1.4661
0.1076	10	.1074	.9942	.1080	9.2553	50	1.4632
0.1105	20	.1103	.9939	.1110	9.0098	40	1.4603
0.1134	30	.1132	.9936	.1139	8.7769	30	1.4574
0.1164	40	.1161	.9932	.1169	8.5555	20	1.4544
0.1193	50	.1190	.9929	.1198	8.3450	10	1.4515
0.1222	7° 00′	.1219	.9925	.1228	8.1443	83° 00′	1.4486
0.1251	10	.1248	.9922	.1257	7.9530	50	1.4457
0.1280	20	.1276	.9918	.1287	7.7704	40	1.4428
0.1309	30	.1305	.9914	.1317	7.5958	30	1.4399
0.1338	40	.1334	.9911	.1346	7.4287	20	1.4370
0.1367	50	.1363	.9907	.1376	7.2687	10	1.4341
0.1396	8° 00′	.1392	.9903	.1405	7.1154	82° 00′	1.4312
0.1425	10	.1421	.9899	.1435	6.9682	50	1.4283
0.1454	20	.1449	.9894	.1465	6.8269	40	1.4254
0.1484	30	.1478	.9890	.1495	6.6912	30	1.4224
0.1513	40	.1507	.9886	.1524	6.5606	20	1.4195
0.1542	50	.1536	.9881	.1554	6.4348	10	1.4166
0.5171	9° 00′	.1564	.9877	.1584	6.3138	81° 00′	1.4137
Radians	Degrees	Cosines	Sines	Cotangents	Tangents	Degrees	Radians

Radians	Degrees	Sines	Cosines	Tangents	Cotangents	Degrees	Radians
0.1571	9° 00′	.1564	.9877	.1584	6.3138	81° 00′	1.4137
0.1600	10	.1593	.9872	.1614	6.1970	50	1.4108
0.1629	20	.1622	.9868	.1644	6.0844	40	1.4097
0.1658	30	.1650	.9863	.1673	5.9758	30	1.4050
0.1687	40	.1679	.9858	.1703	5.8708	20	1.4021
0.1716	50	.1708	.9853	.1733	5.7694	10	1.3992
0.1745	10° 00′	.1736	.9848	.1763	5.6713	80° 00′	1.3963
0.1774	10	.1765	.9843	.1793	5.5764	50	1.3934
0.1804	20	.1794	.9838	.1823	5.4845	40	1.3904
0.1833	30	.1822	.9833	.1853	5.3955	30	1.3875
0.1862	40	.1851	.9827	.1883	5.3093	20	1.3846
0.1891	50	.1880	.9822	.1914	5.2257	10	1.3817
0.1920	11° 00′	.1908	.9816	.1944	5.1446	79° 00′	1.3788
0.1949	10	.1937	.9811	.1974	5.0658	50	1.3759
0.1978	20	.1965	.9805	.2004	4.9894	40	1.3730
0.2007	30	.1994	.9799	.2035	4.9152	30	1.3701
0.2036	40	.2022	.9793	.2065	4.8430	20	1.3672
0.2065	50	.2051	.9787	.2095	4.7729	10	1.3643
0.2094	12° 00′	.2079	.9781	.2126	4.7046	78° 00′	1.3614
0.2123	10	.2108	.9775	.2156	4.6382	50	1.3584
0.2153	20	.2136	.9769	.2186	4.5736	40	1.3555
0.2182	30	.2164	.9763	.2217	4.5107	30	1.3526
0.2211	40	.2193	.9757	.2247	4.4494	20	1.3497
0.2240	50	.2221	.9750	.2278	4.3897	10	1.3468
0.2269	13° 00′	.2250	.9744	.2309	4.3315	77° 00′	1.3439
0.2298	10	.2278	.9737	.2339	4.2747	50	1.3410
0.2327	20	.2306	.9730	.2370	4.2193	40	1.3381
0.2356	30	.2334	.9724	.2401	4.1653	30	1.3352
0.2385	40	.2363	.9717	.2432	4.1126	20	1.3323
0.2414	50	.2391	.9710	.2462	4.0611	10	1.3294
0.2443	14° 00′	.2419	.9703	.2493	4.0108	76° 00′	1.3265
0.2473	10	.2447	.9696	.2524	3.9617	50	1.3235
0.2502	20	.2476	.9689	.2555	3.9136	40	1.3206
0.2531	30	.2504	.9681	.2586	3.8667	30	1.3177
0.2560	40	.2532	.9674	.2617	3.8208	20	1.3148
0.2589	50	.2560	.9667	.2648	3.7760	10	1.3119
0.2618	15° 00′	.2588	.9659	.2679	3.7321	75° 00′	1.3090
0.2647	10	.2616	.9652	.2711	3.6891	50	1.3061
0.2676	20	.2644	.9644	.2742	3.6470	40	1.3032
0.2705	30	.2672	.9636	.2773	3.6059	30	1.3003
0.2734	40	.2700	.9628	.2805	3.5656	20	1.2974
0.2763	50	.2728	.9621	.2836	3.5261	10	1.2945
0.2793	16° 00′	.2756	.9613	.2867	3.4874	74° 00′	1.2915
0.2822	10	.2784	.9605	.2899	3.4495	50	1.2886
0.2851	20	.2812	.9596	.2931	3.4124	40	1.2857
0.2880	30	.2840	.9588	.2962	3.3759	30	1.2828
0.2909	40	.2868	.9580	.2994	3.3402	20	1.2799
0.2938	50	.2896	.9572	.3026	3.3052	10	1.2770
0.2967	17° 00′	.2924	.9563	.3057	3.2709	73° 00′	1.2741
0.2996	10	.2952	.9555	.3089	3.2371	50	1.2712
0.3025	20	.2979	.9546	.3121	3.2041	40	1.2683
0.3054	30	.3007	.9537	.3153	3.1716	30	1.2654
0.3083	40	.3035	.9528	.3185	3.1397	20	1.2625
0.3113	50	.3062	.9520	.3217	3.1084	10	1.2595
0.3142	18° 00′	.3090	.9511	.3249	3.0777	72° 00′	1.2566
Radians	Degrees	Cosines	Sines	Cotangents	Tangents	Degrees	Radians

Radians	Degrees	Sines	Cosines	Tangents	Cotangents	Degrees	Radians
0.3142	18° 00′	.3090	.9511	.3249	3.0777	72° 00′	1.2566
0.3171	10	.3118	.9502	.3281	3.0475	50	1.2537
0.3200	20	.3145	.9492	.3314	3.0178	40	1.2508
0.3229	30	.3173	.9483	.3346	2.9887	30	1.2479
0.3258	40	.3201	.9474	.3378	2.9600	20	1.2450
0.3287	50	.3228	.9465	.3411	2.9319	10	1.2421
0.3316	19° 00′	.3256	.9455	.3443	2.9042	71° 00′	1.2392
0.3345	10	.3283	.9446	.3476	2.8770	50	1.2363
0.3374	20	.3311	.9436	.3508	2.8502	40	1.2334
0.3403	30	.3338	.9426	.3541	2.8239	30	1.2305
0.3432	40	.3365	.9417	.3574	2.7980	20	1.2275
0.3462	50	.3393	.9407	.3607	2.7725	10	1.2246
0.3491	20° 00′	.3420	.9397	.3640	2.7475	70° 00′	1.2217
0.3520	10	.3448	.9387	.3673	2.7228	50	1.2188
0.3549	20	.3475	.9377	.3706	2.6985	40	1.2159
0.3578	30	.3502	.9367	.3739	2.6746	30	1.2130
0.3607	40	.3529	.9356	.3772	2.6511	20	1.2101
0.3636	50	.3557	.9346	.3805	2.6279	10	1.2072
0.3665	21° 00′	.3584	.9336	.3839	2.6051	69° 00′	1.2043
0.3694	10	.3611	.9325	.3872	2.5826	50	1.2014
0.3723	20	.3638	.9315	.3906	2.5605	40	1.1985
0.3752	30	.3665	.9304	.3939	2.5386	30	1.1956
0.3782	40	.3692	.9293	.3973	2.5172	20	1.1926
0.3811	50	.3719	.9283	.4006	2.4960	10	1.1897
0.3840	22° 00′	.3746	.9272	.4040	2.4751	68° 00′	1.1868
0.3869	10	.3773	.9261	.4074	2.4545	50	1.1839
0.3898	20	.3800	.9250	.4108	2.4342	40	1.1810
0.3927	30	.3827	.9239	.4142	2.4142	30	1.1781
0.3956	40	.3854	.9228	.4176	2.3945	20	1.1752
0.3985	50	.3881	.9216	.4210	2.3750	10	1.1723
0.4014	23° 00′	.3907	.9205	.4245	2.3559	67° 00′	1.1694
0.4043	10	.3934	.9194	.4279	2.3369	50	1.1665
0.4072	20	.3961	.9182	.4314	2.3183	40	1.1636
0.4102	30	.3987	.9171	.4348	2.2998	30	1.1606
0.4131	40	.4014	.9159	.4383	2.2817	20	1.1577
0.4160	50	.4041	.9147	.4417	2.2637	10	1.1548
0.4189	24° 00′	.4067	.9135	.4452	2.2460	66° 00′	1.1519
0.4218	10	.4094	.9124	.4487	2.2286	50	1.1490
0.4247	20	.4120	.9112	.4522	2.2113	40	1.1461
0.4276	30	.4147	.9100	.4557	2.1943	30	1.1432
0.4305	40	.4173	.9088	.4592	2.1775	20	1.1403
0.4334	50	.4200	.9075	.4628	2.1609	10	1.1374
0.4363	25° 00′	.4226	.9063	.4663	2.1445	65° 00′	1.1345
0.4392	10	.4253	.9051	.4699	2.1283	50	1.1316
0.4422	20	.4279	.9038	.4734	2.1123	40	1.1286
0.4451	30	.4305	.9026	.4770	2.0965	30	1.1257
0.4480	40	.4331	.9013	.4806	2.0809	20	1.1228
0.4509	50	.4358	.9001	.4841	2.0655	10	1.1199
0.4538	26° 00′	.4384	.8988	.4877	2.0503	64° 00′	1.1170
0.4567	10	.4410	.8975	.4913	2.0353	50	1.1141
0.4596	20	.4436	.8962	.4950	2.0204	40	1.1112
0.4625	30	.4462	.8949	.4986	2.0057	30	1.1083
0.4654	40	.4488	.8936	.5022	1.9912	20	1.1054
0.4683	50	.4514	.8923	.5059	1.9768	10	1.1025
0.4712	27° 00′	.4540	.8910	.5095	1.9626	63° 00′	1.0996
Radians	Degrees	Cosines	Sines	Cotangents	Tangents	Degrees	Radians

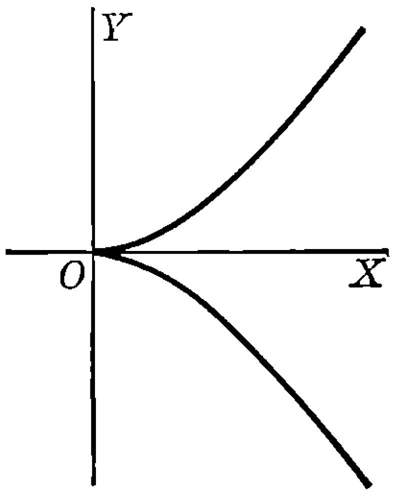
Radians	Degrees	Sines	Cosines	Tangents	Cotangents	Degrees	Radians
0.4712	27° 00′	.4540	.8910	.5095	1.9626	63° 00′	1.0996
0.4741	10	.4566	.8897	.5132	1.9486	50	1.0966
0.4771	20	.4592	.8884	.5169	1.9347	40	1.0937
0.4800	30	.4617	.8870	.5206	1.9210	30	1.0908
0.4829	40	.4643	.8857	.5243	1.9074	20	1.0879
0.4858	50	.4669	.8843	.5280	1.8940	10	1.0850
0.4887	28° 00′	.4695	.8829	.5317	1.8807	62° 00′	1.0821
0.4916	10	.4720	.8816	.5354	1.8676	50	1.0792
0.4945	20	.4746	.8802	.5392	1.8546	40	1.0763
0.4974	30	.4772	.8788	.5430	1.8418	30	1.0734
0.5003	40	.4797	.8774	.5467	1.8291	20	1.0705
0.5032	50	.4823	.8760	.5505	1.8165	10	1.0676
0.5061	29° 00′	.4848	.8746	.5543	1.8040	61° 00′	1.0647
0.5091	10	.4874	.8732	.5581	1.7917	50	1.0617
0.5120	20	.4899	.8718	.5619	1.7796	40	1.0588
0.5149	30	.4924	.8704	.5658	1.7675	30	1.0559
0.5178	40	.4950	.8689	.5696	1.7556	20	1.0530
0.5207	50	.4975	.8675	.5735	1.7437	10	1.0501
0.5236	30° 00′	.5000	.8660	.5774	1.7321	60° 00′	1.0472
0.5265	10	.5025	.8646	.5812	1.7205	50	1.0443
0.5294	20	.5050	.8631	.5851	1.7090	40	1.0414
0.5323	30	.5075	.8616	.5890	1.6977	30	1.0385
0.5352	40	.5100	.8601	.5930	1.6864	20	1.0356
0.5381	50	.5125	.8587	.5969	1.6753	10	1.0327
0.5411	31° 00′	.5150	.8572	.6009	1.6643	59° 00′	1.0297
0.5440	10	.5175	.8557	.6048	1.6534	50	1.0268
0.5469	20	.5200	.8542	.6088	1.6426	40	1.0239
0.5498	30	.5225	.8526	.6128	1.6319	30	1.0210
0.5527	40	.5250	.8511	.6168	1.6212	20	1.0181
0.5556	50	.5275	.8496	.6208	1.6107	10	1.0152
0.5585	32° 00′	.5299	.8480	.6249	1.6003	58° 00′	1.0123
0.5614	10	.5324	.8465	.6289	1.5900	50	1.0094
0.5643	20	.5348	.8450	.6330	1.5798	40	1.0065
0.5672	30	.5373	.8434	.6371	1.5697	30	1.0036
0.5701	40	.5398	.8418	.6412	1.5597	20	1.0007
0.5730	50	.5422	.8403	.6453	1.5497	10	0.9977
0.5760	33° 00′	.5446	.8387	.6494	1.5399	57° 00′	0.9948
0.5789	10	.5471	.8371	.6536	1.5301	50	0.9919
0.5818	20	.5495	.8355	.6577	1.5204	40	0.9890
0.5847	30	.5519	.8339	.6619	1.5108	30	0.9861
0.5876	40	.5544	.8323	.6661	1.5013	20	0.9832
0.5905	50	.5568	.8307	.6703	1.4919	10	0.9803
0.5934	34° 00′	.5592	.8290	.6745	1.4826	56° 00′	0.9774
0.5963	10	.5616	.8274	.6787	1.4733	50	0.9745
0.5992	20	.5640	.8258	.6830	1.4641	40	0.9716
0.6021	30	.5664	.8241	.6873	1.4550	30	0.9687
0.6050	40	.5688	.8225	.6916	1.4460	20	0.9657
0.6080	50	.5712	.8208	.6959	1.4370	10	0.9628
0.6109	35° 00′	.5736	.8192	.7002	1.4281	55° 00′	0.9599
0.6138	10	.5760	.8175	.7046	1.4193	50	0.9570
0.6167	20	.5783	.8158	.7089	1.4106	40	0.9541
0.6196	30	.5807	.8141	.7133	1.4019	30	0.9512
0.6225	40	.5831	.8124	.7177	1.3934	20	0.9483
0.6254	50	.5854	.8107	.7221	1.3848	10	0.9454
0.6283	36° 00′	.5878	.8090	.7265	1.3764	54° 00′	0.9425
Radians	Degrees	Cosines	Sines	Cotangents	Tangents	Degrees	Radians

Radians	Degrees	Sines	Cosines	Tangents	Cotangents	Degrees	Radians
0.6283	36° 00′	.5878	.8090	.7265	1.3764	54° 00′	0.9425
0.6312	10	.5901	.8073	.7310	1.3680	50	0.9396
0.6341	20	.5925	.8056	.7355	1.3597	40	0.9367
0.6370	30	.5948	.8039	.7400	1.3514	30	0.9338
0.6400	40	.5972	.8021	.7445	1.3432	20	0.9308
0.6429	50	.5995	.8004	.7490	1.3351	10	0.9279
0.6458	37° 00′	.6018	.7986	.7536	1.3270	53° 00′	0.9250
0.6487	10	.6041	.7969	.7581	1.3190	50	0.9221
0.6516	20	.6065	.7951	.7627	1.3111	40	0.9192
0.6545	30	.6088	.7934	.7673	1.3032	30	0.9163
0.6574	40	.6111	.7916	.7720	1.2954	20	0.9134
0.6603	50	.6134	.7898	.7766	1.2876	10	0.9105
0.6632	38° 00′	.6157	.7880	.7813	1.2799	52° 00′	0.9076
0.6661	10	.6180	.7862	.7860	1.2723	50	0.9047
0.6690	20	.6202	.7844	.7907	1.2647	40	0.9018
0.6720	30	.6225	.7826	.7954	1.2572	30	0.8988
0.6749	40	.6248	.7808	.8002	1.2497	20	0.8959
0.6778	50	.6271	.7790	.8050	1.2423	10	0.8930
0.6807	39° 00′	.6293	.7771	.8098	1.2349	51° 00′	0.8901
0.6836	10	.6316	.7753	.8146	1.2276	50	0.8872
0.6865	20	.6338	.7735	.8195	1.2203	40	0.8843
0.6894	30	.6361	.7716	.8243	1.2131	30	0.8814
0.6923	40	.6383	.7698	.8292	1.2059	20	0.8785
0.6952	50	.6406	.7679	.8342	1.1988	10	0.8756
0.6981	40° 00′	.6428	.7660	.8391	1.1918	50° 00′	0.8727
0.7010	10	.6450	.7642	.8441	1.1847	50	0.8698
0.7039	20	.6472	.7623	.8491	1.1778	40	0.8668
0.7069	30	.6494	.7604	.8541	1.1708	30	0.8639
0.7098	40	.6517	.7585	.8591	1.1640	20	0.8610
0.7127	50	.6539	.7566	.8642	1.1571	10	0.8581
0.7156	41° 00′	.6561	.7547	.8693	1.1504	49° 00′	0.8552
0.7185	10	.6583	.7528	.8744	1.1436	50	0.8523
0.7214	20	.6604	.7509	.8796	1.1369	40	0.8494
0.7243	30	.6626	.7490	.8847	1.1303	30	0.8465
0.7272	40	.6648	.7470	.8899	1.1237	20	0.8436
0.7301	50	.6670	.7451	.8952	1.1171	10	0.8407
0.7330	42° 00′	.6691	.7431	.9004	1.1106	48° 00′	0.8378
0.7359	10	.6713	.7412	.9057	1.1041	50	0.8348
0.7389	20	.6734	.7392	.9110	1.0977	40	0.8319
0.7418	30	.6756	.7373	.9163	1.0913	30	0.8290
0.7447	40	.6777	.7353	.9217	1.0850	20	0.8261
0.7476	50	.6799	.7333	.9271	1.0786	10	0.8232
0.7505	43° 00′	.6820	.7314	.9325	1.0724	47° 00′	0.8203
0.7534	10	.6841	.7294	.9380	1.0661	50	0.8174
0.7563	20	.6862	.7274	.9435	1.0599	40	0.8145
0.7592	30	.6884	.7254	.9490	1.0538	30	0.8116
0.7621	40	.6905	.7234	.9545	1.0477	20	0.8087
0.7650	50	.6926	.7214	.9601	1.0416	10	0.8058
0.7679	44° 00′	.6947	.7193	.9657	1.0355	46° 00′	0.8029
0.7709	10	.6967	.7173	.9713	1.0295	50	0.7999
0.7738	20	.6988	.7153	.9770	1.0235	40	0.7970
0.7767	30	.7009	.7133	.9827	1.0176	30	0.7941
0.7796	40	.7030	.7112	.9884	1.0117	20	0.7912
0.7825	50	.7050	.7092	.9942	1.0058	10	0.7883
0.7854	45° 00′	.7071	.7071	1.0000	1.0000	45° 00′	0.7854
Radians	Degrees	Cosines	Sines	Cotangents	Tangents	Degrees	Radians

VII. SQUARES AND CUBES, SQUARE ROOTS AND CUBE ROOTS, AND RECIPROCALS OF INTEGERS FROM 1 TO 100

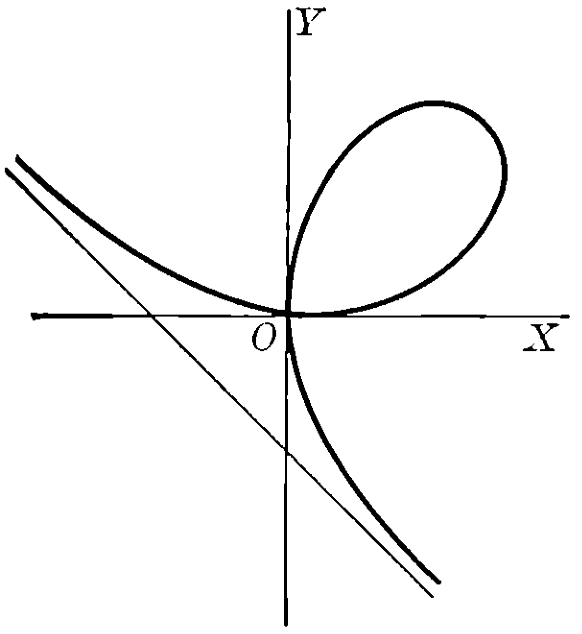
n	Square n^2	Cube n^3	Square Root $\sqrt{n}$	Cube Root $\sqrt[3]{n}$	Reciprocal $1/n$	n	Square n^2	Cube n^3	Square Root $\sqrt{n}$	Cube Root $\sqrt[3]{n}$	Reciprocal $1/n$
1	1	1	1.000	1.000	1.0000	**51**	2,601	132,651	7.141	3.708	.0196
2	4	8	1.414	1.260	.5000	**52**	2,704	140,608	7.211	3.733	.0192
3	9	27	1.732	1.442	.3333	**53**	2,809	148,877	7.280	3.756	.0189
4	16	64	2.000	1.587	.2500	**54**	2,916	157,464	7.348	3.780	.0185
5	25	125	2.236	1.710	.2000	**55**	3,025	166,375	7.416	3.803	.0182
6	36	216	2.449	1.817	.1667	**56**	3,136	175,616	7.483	3.826	.0179
7	49	343	2.646	1.913	.1429	**57**	3,249	185,193	7.550	3.849	.0175
8	64	512	2.828	2.000	.1250	**58**	3,364	195,112	7.616	3.871	.0172
9	81	729	3.000	2.080	.1111	**59**	3,481	205,379	7.681	3.893	.0170
10	100	1,000	3.162	2.154	.1000	**60**	3,600	216,000	7.746	3.915	.0167
11	121	1,331	3.317	2.224	.0909	**61**	3,721	226,981	7.810	3.936	.0164
12	144	1,728	3.464	2.289	.0833	**62**	3,844	238,328	7.874	3.958	.0161
13	169	2,197	3.606	2.351	.0769	**63**	3,969	250,047	7.937	3.979	.0159
14	196	2,744	3.742	2.410	.0714	**64**	4,096	262,144	8.000	4.000	.0156
15	225	3,375	3.873	2.466	.0667	**65**	4,225	274,625	8.062	4.021	.0154
16	256	4,096	4.000	2.520	.0625	**66**	4,356	287,496	8.124	4.041	.0152
17	289	4,913	4.123	2.571	.0588	**67**	4,489	300,763	8.185	4.062	.0149
18	324	5,832	4.243	2.621	.0556	**68**	4,624	314,432	8.246	4.082	.0147
19	361	6,859	4.359	2.668	.0526	**69**	4,761	328,509	8.307	4.102	.0145
20	400	8,000	4.472	2.714	.0500	**70**	4,900	343,000	8.367	4.121	.0143
21	441	9,261	4.583	2.759	.0476	**71**	5,041	357,911	8.426	4.141	.0141
22	484	10,648	4.690	2.802	.0455	**72**	5,184	373,248	8.485	4.160	.0139
23	529	12,167	4.796	2.844	.0436	**73**	5,329	389,017	8.544	4.179	.0137
24	576	13,824	4.899	2.884	.0417	**74**	5,476	405,224	8.602	4.198	.0135
25	625	15,625	5.000	2.924	.0400	**75**	5,625	421,875	8.660	4.217	.0133
26	676	17,576	5.099	2.962	.0385	**76**	5,776	438,976	8.718	4.236	.0132
27	729	19,683	5.196	3.000	.0370	**77**	5,929	456,533	8.775	4.254	.0130
28	784	21,952	5.292	3.037	.0357	**78**	6,084	474,552	8.832	4.273	.0128
29	841	24,389	5.385	3.072	.0345	**79**	6,241	493,039	8.888	4.291	.0127
30	900	27,000	5.477	3.107	.0333	**80**	6,400	512,000	8.944	4.309	.0125
31	961	29,791	5.568	3.141	.0323	**81**	6,561	531,441	9.000	4.327	.0123
32	1,024	32,768	5.657	3.175	.0313	**82**	6,724	551,368	9.055	4.344	.0122
33	1,089	35,937	5.745	3.208	.0303	**83**	6,889	571,787	9.110	4.362	.0120
34	1,156	39,304	5.831	3.240	.0294	**84**	7,056	592,704	9.165	4.380	.0119
35	1,225	42,875	5.916	3.271	.0286	**85**	7,225	614,125	9.220	4.397	.0118
36	1,296	46,656	6.000	3.302	.0278	**86**	7,396	636,056	9.274	4.414	.0116
37	1,369	50,653	6.083	3.332	.0270	**87**	7,569	658,503	9.327	4.431	.0115
38	1,444	54,872	6.164	3.362	.0263	**88**	7,744	681,472	9.381	4.448	.0114
39	1,521	59,319	6.245	3.391	.0256	**89**	7,921	704,969	9.434	4.465	.0112
40	1,600	64,000	6.325	3.420	.0250	**90**	8,100	729,000	9.487	4.481	.0111
41	1,681	68,921	6.403	3.448	.0244	**91**	8,281	753,571	9.539	4.498	.0110
42	1,764	74,088	6.481	3.476	.0238	**92**	8,464	778,688	9.592	4.514	.0109
43	1,849	79,507	6.557	3.503	.0233	**93**	8,649	804,357	9.644	4.531	.0108
44	1,936	85,184	6.633	3.530	.0227	**94**	8,836	830,584	9.695	4.547	.0106
45	2,025	91,125	6.708	3.557	.0222	**95**	9,025	857,375	9.747	4.563	.0105
46	2,116	97,336	6.782	3.583	.0217	**96**	9,216	884,736	9.798	4.579	.0104
47	2,209	103,823	6.856	3.609	.0213	**97**	9,409	912,673	9.849	4.595	.0103
48	2,304	110,592	6.928	3.634	.0208	**98**	9,604	941,192	9.899	4.610	.0102
49	2,401	117,649	7.000	3.659	.0204	**99**	9,801	970,299	9.950	4.626	.0101
50	2,500	125,000	7.071	3.684	.0200	**100**	10,000	1,000,000	10.000	4.642	.0100
n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$

Semicubical Parabola



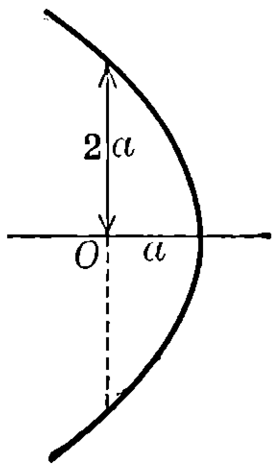
$y^2 = ax^3$

Folium



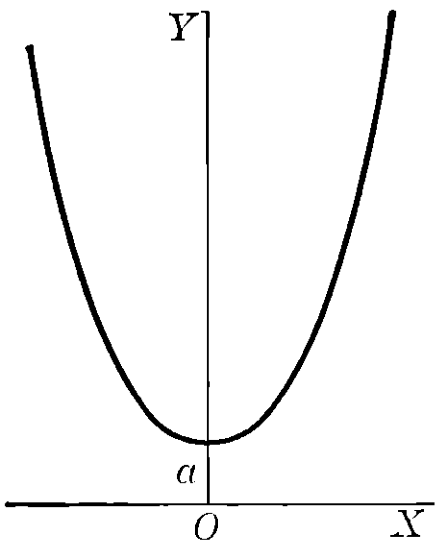
$x^3 + y^3 = 3\,axy$

Parabola



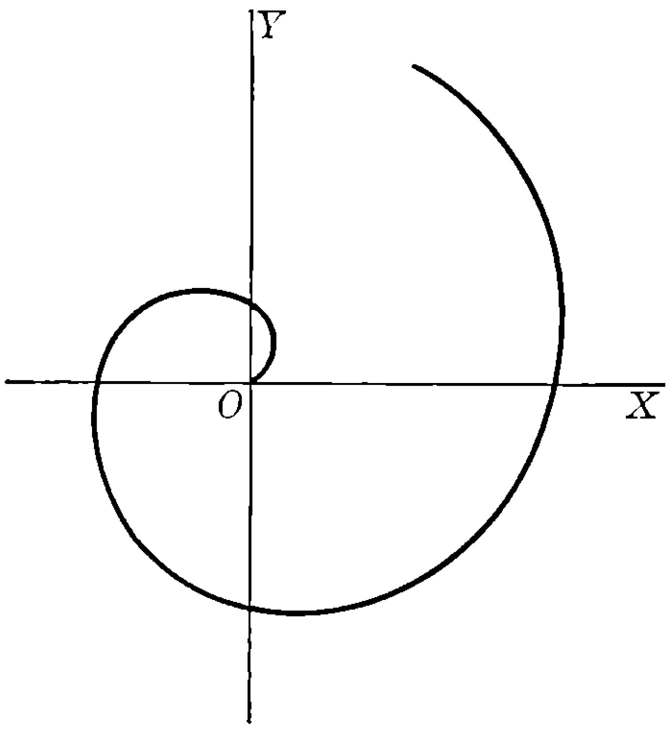
$\rho = a \sec^2 \frac{\theta}{2}$

Catenary



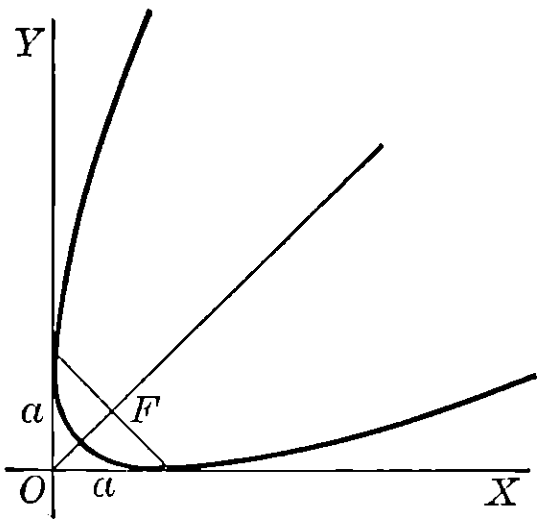
$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$

Spiral of Archimedes



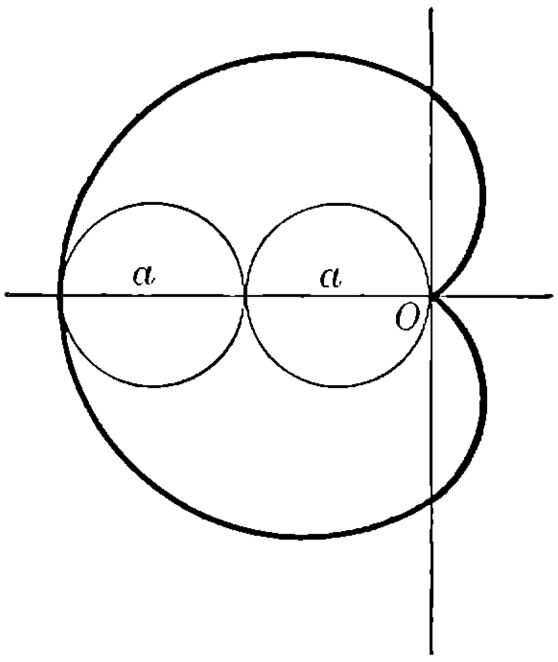
$\rho = a\theta$

Parabola



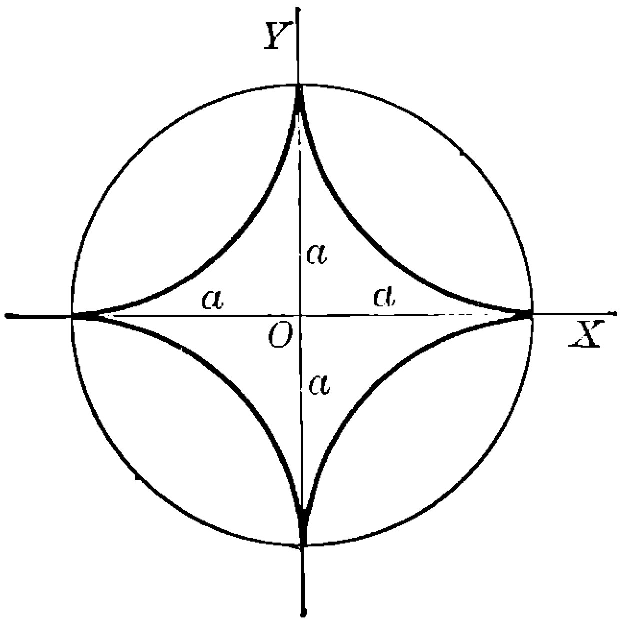
$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

Cardioid



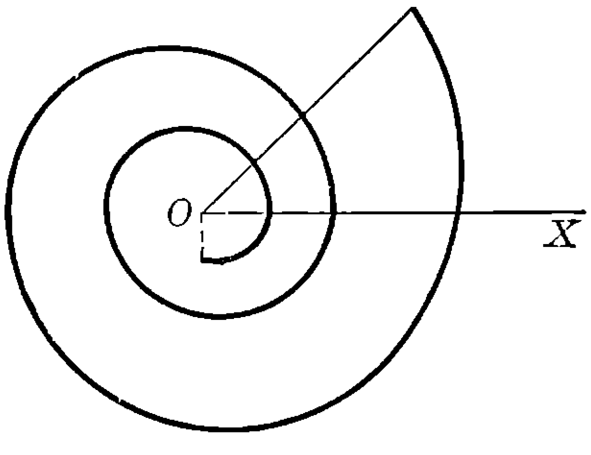
$\rho = a(1 - \cos\theta)$

Four-Cusped Hypocycloid



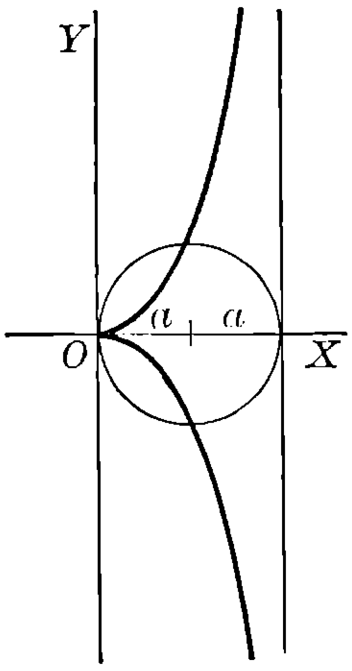
$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Logarithmic Spiral



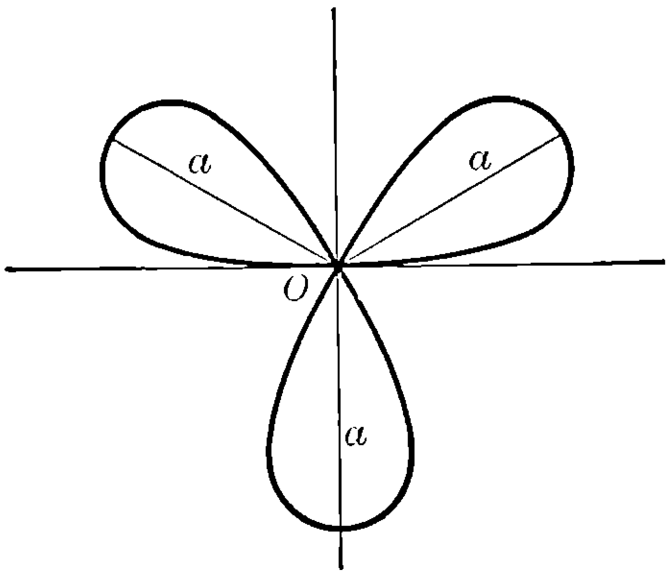
$\log \rho = a\theta$

Cissoid



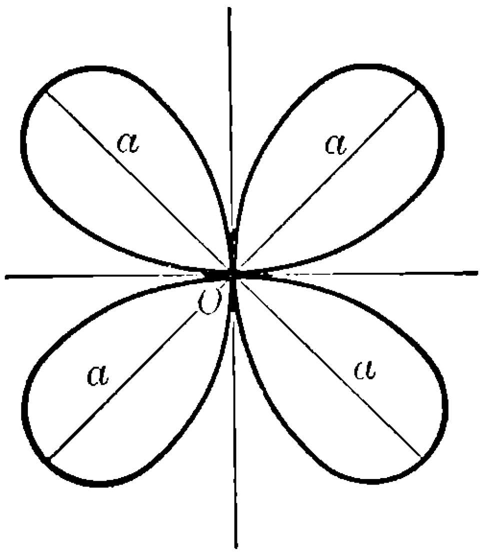
$y^2(2a - x) = x^3$

Three-Leafed Rose

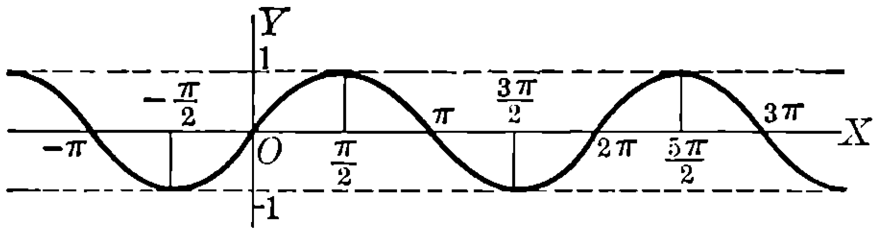


$\rho = a \sin 3\theta$

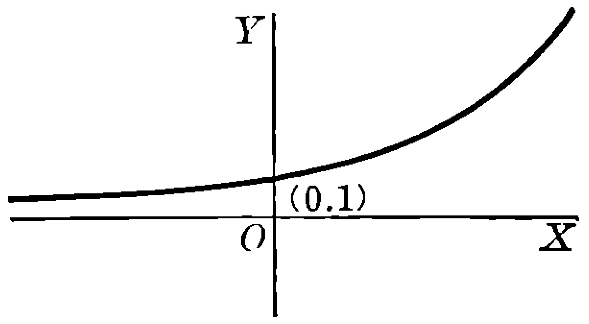
Four-Leafed Rose



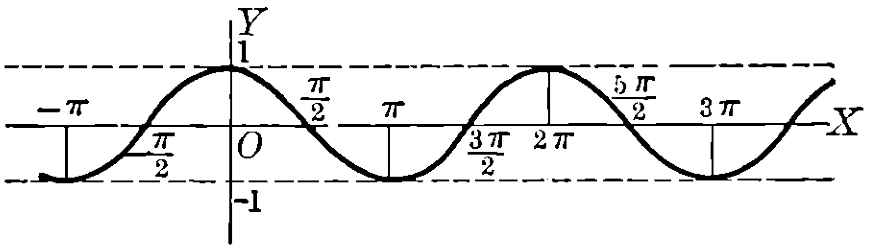
$\rho = a \sin 2\theta$



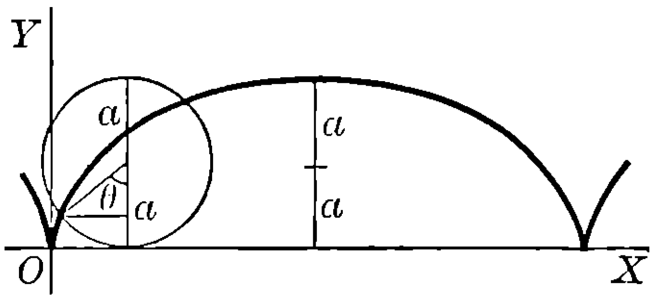
SINUSOID. $y = \sin x$



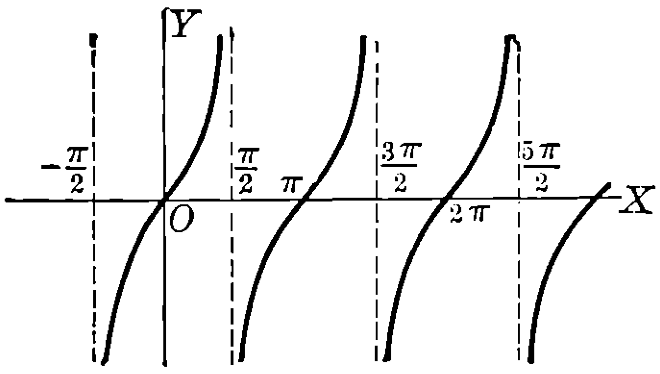
EXPONENTIAL CURVE. $y = a^x$



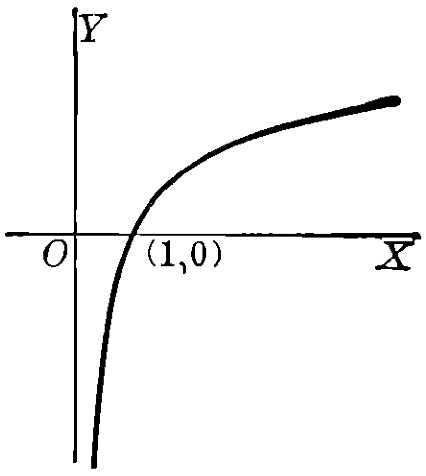
$y = \cos x$



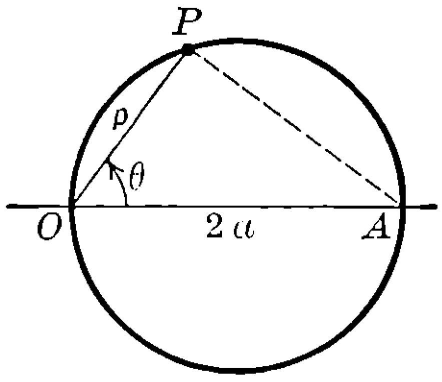
CYCLOID. $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$



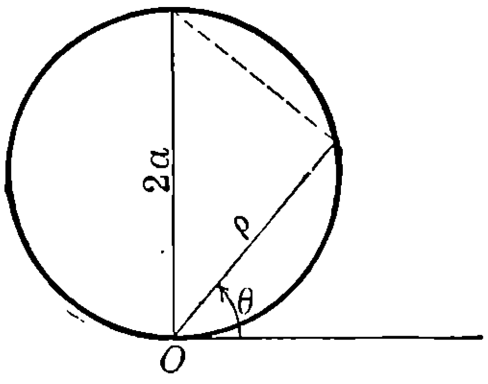
$y = \tan x$



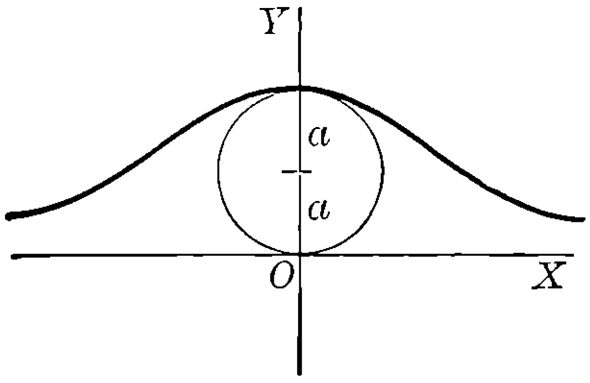
LOGARITHMIC CURVE. $y = \log_a x$



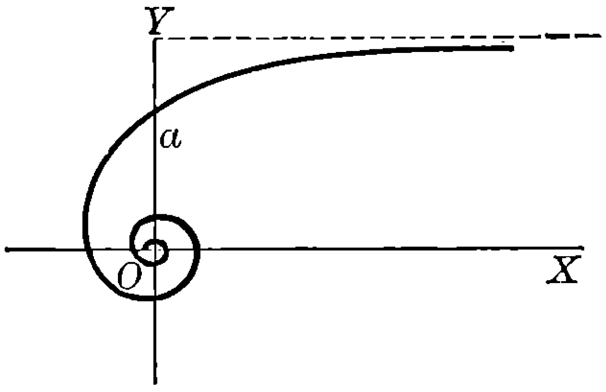
CIRCLE. $\rho = 2a\cos\theta$



CIRCLE. $\rho = 2a\sin\theta$



WITCH. $x^2y = 4a^2(2a - y)$



HYPERBOLIC SPIRAL. $\rho\theta = a$

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